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SUNNYVALE, CALIFORNIA

HREC-1146-2
LMSC/HREC D149128

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HUNTSVILLE RESEARCH & ENGINEERING CENTER

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4800 BRADFORD BLVD., HUNTSVILLE, ALABAMA

OPTIMUM RENDEZVOUS GUIDANCE STUDY

FINAL REPORT

August 1969

Contract NAS8-21146

by

Joerg Waldvogel

T. R. Beal, Manager
Dynamics & Guidance Dept.

J. S. Farrior
Resident Director

FOREWORD

This report presents the results of work performed by Lockheed's Huntsville Research & Engineering Center, under Contract NAS8-21146 for the Aero-Astrodynamics Laboratory of the NASA/Marshall Space Flight Center.

It is the last of a series of reports prepared by Lockheed from 1966 through 1969 under Contract NAS8-18036 and under the above mentioned contract.

The NASA technical coordinator for the present study is Mr. Roger R. Burrows, S&E-AERO-GG.

SUMMARY

This study considers the problem of flight scheduling in the planar two-burn minimum fuel rendezvous of an interceptor with a target vehicle. A set of equations is developed which takes position and velocity of the interceptor and of the target as input data. These equations allow calculation of five control parameters: the durations of the first burn, of the coast, and of the second burn, and the average thrust direction for each burn period.

In order to set up the rendezvous conditions, Levi-Civita's regularized variables and the corresponding orbital elements are used in contrast to most other papers on rendezvous problems. This brings several advantages compared to the use of polar coordinates:

- An elliptic target orbit can readily be handled.
- A near-circular coast orbit of the interceptor does not cause difficulties because the (badly defined) periapsis is not used.
- The resulting equations are fairly simple and their numerical treatment is stable.

The scheme uses some simplifying assumptions which, however, are satisfied in most practical cases:

- The burn durations are assumed to be short compared to the duration of the coast.
- The Keplerian ellipses involved must have small eccentricities.
- Instead of dealing with a variable thrust vector, the scheme uses a constant average value during each burn.

Due to these simplifications, the scheme will furnish values of the control parameters which generally do not result in an exact rendezvous of the two vehicles.

However, updating these control parameters throughout the first burn yields a more accurate rendezvous after the coast and the second burn.

Thus, due to the simplicity of the equations (resulting in short computation time) the scheme can also operate as a first-burn guidance scheme in the sense that the flight scheduling is done repeatedly based on current position and velocity data.

In order to accomplish the rendezvous accurately by the second burn, a closed loop terminal guidance scheme based on measurements of the relative position and velocity is necessary. The approximations and simplifications in the present scheme are too rough for this purpose.

Such a scheme, the Dual Phase Plane Method, has been derived and simulated in Ref. 3. Hence, in this report rendezvous missions are only simulated up to the end of the coast, where the terminal guidance scheme can take over. The second burn is handled without updating according to the latest values of the control parameters calculated at the end of the first burn.

Simulation results for a typical rendezvous case are included in this report.

CONTENTS

NOMENCLATURE

a	semi-major axis
α_j, β_j	regularized elements
γ	flight path angle
D	total burn duration
D_0, D_2	first and second burn duration
$\Delta\alpha_j$, $\Delta\beta_j$	element increments
δ	separation angle
δ_{jk}	Kronecker symbol
e	exit velocity of the thruster
F_j	perturbation function
f	thrust acceleration
I	interceptor
μ	earth's gravitational parameter
O	earth's center, origin
p_j	perturbing forces
q_j	generalized forces
r	distance from the origin
s	regularized time, value for the interceptor at rendezvous
s_0, s_2	increments of s during first, second burn
σ	value of s for the target at rendezvous
T	target
t	time

NOMENCLATURE (Continued)

t_{coa}	coast duration
t_{tar}	mission duration
τ	"burn-out duration" = initial mass of I divided by mass flow rate
u_j	Levi-Civita's variables
v	velocity
w_j	modified derivatives of the u
ω	frequency
x_j	Cartesian coordinates
χ	thrust angle (measured from the fixed x_1-direction)
Z	side condition
$\Delta\gamma_j$, G_0, G_2, λ, M, φ	auxiliary quantities

Subscripts

j = 1, 2

The subscripts with the following meanings are mostly omitted in the preceding list. In the text, however, they are used in addition to the subscripts appearing already here.

0	initial values of the interceptor, first burn
1	coast
2	rendezvous, second burn
3	initial values of the target

Section 1
INTRODUCTION

Optimum rendezvous problems are boundary value problems for differential equations (DEQ) where a certain cost function must be minimized by an appropriate choice of control functions. By the modern methods of the calculus of variations (for instance Pontryagin's principle, Ref. 1) it is possible to solve problems of this kind exactly, but only with a considerable computational effort.

The goal of this study is to simplify the rendezvous problem in an appropriate way, such that the results can be obtained within seconds by an onboard computer, but without losing too much accuracy (5 km position error).

The way to obtain simplifications is to introduce restrictions which are satisfied in most practical cases:

- The interceptor's trajectory is assumed to lie in a narrow circular ring.

Then the reouired velocity increments are rather small and can be attained by

- short burn durations.

This allows linearization with respect to these burn durations.

- The thrust forces are considered as perturbing forces acting on the interceptor, I, whose unperturbed orbit is a Kepler ellipse. Only first order perturbations are considered.
- During each of the short burns the thrust is put constant in magnitude and direction.

This reduces the problem of calculus of variations to an ordinary minimum problem.

Even if all these simplifications are made the system of equations to be solved is quite complicated. Much depends upon the choice of the coordinates for describing the trajectory in the powered flight phases and upon the orbital elements used for characterizing the coast periods. Three possibilities are considered:

1. Polar Coordinates Associated with Classical Orbital Elements

This choice is striking because of the simple geometric meaning of the polar coordinates and the classical elements. Unfortunately this method fails in many cases we are concerned with, since the classical elements are badly defined for near-circular orbits (the perigee for a circular orbit is undefined). This approach has been the subject of earlier publications (Refs. 2, 3). In these reports a very efficient terminal guidance technique, the Dual Phase Plane Method due to I. Kliger and W. Trautwein, has also been described (see in particular Refs. 4, 5).

2. Levi-Civita's Regularized Coordinates and the Corresponding Elements

Although the application of this set of parameters yields more complicated equations, it is advantageous due to the "linearizing" effect of Levi-Civita's transformation (see Section 3). Transition through a circular orbit causes no difficulties in these parameters. Most of the present report is concerned with the derivation of the control laws in this case.

3. True Anomaly as Independent Variable

The Kepler motion, described by direction unit vector and reciprocal distance as functions of the true anomaly satisfies a system of linear DEQ with constant coefficients (Ref. 6). Thus, applying these parameters has the advantage of Levi-Civita's variables, yet transformations of that complexity are not used.

Only a short description of the parameters and the corresponding DEQ will be given in Section 6.

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In reply refer to:
LMSC/HREC D149006

18 August 1969

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Very truly yours,

LOCKHEED MISSILES & SPACE COMPANY

James E. Eckert

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Section 2
THE RENDEZVOUS

Let T be a passive target vehicle moving on an elliptic orbit about the earth's center O. In the same orbital plane a steerable interceptor I is assumed to coast on an elliptic parking orbit. A rectangular coordinate system x_1, x_2 centered at O is used in the common orbital plane. The rendezvous mission consists of transferring I to T such that they meet with equal velocities and the least possible amount of fuel is used. The interceptor's engine is supposed to be ignited for the first time at a given time t = 0.

The system of the two vehicles is characterized by the quantity

μ = earth's gravitational parameter

and by two parameters associated with the interceptor's thruster:

τ = initial mass of I divided by the engine's mass flow rate
e = exit velocity.

The situation at time t = 0 is given by the initial target data

r_3 = distance OT_3
v_3 = initial target velocity
γ_3 = target flight path angle

and the initial interceptor data

r_0 = distance OI_0
v_0 = initial interceptor velocity
γ_0 = interceptor flight path angle
δ = initial separation angle according to Fig. 1.

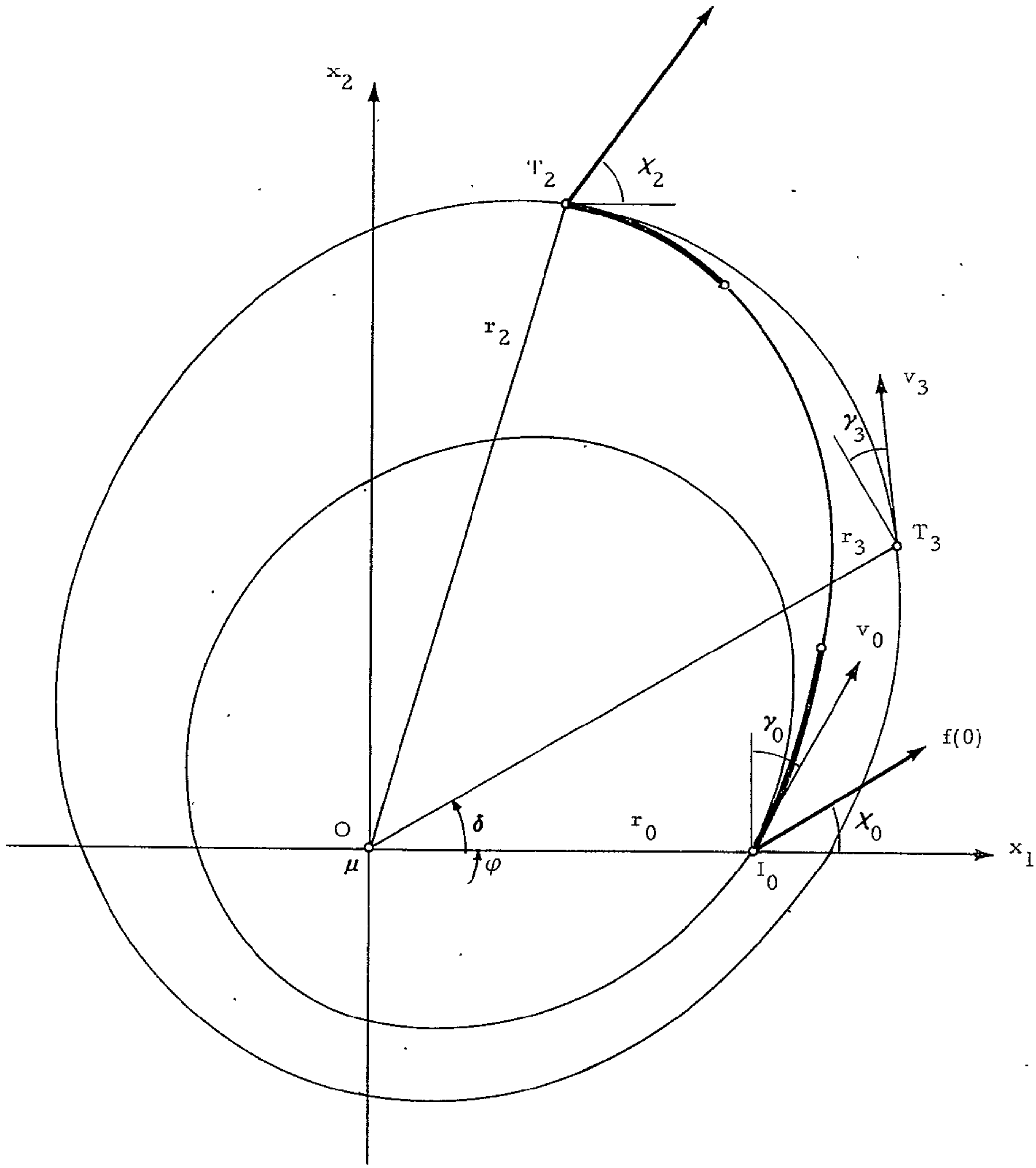

Fig. 1 - Initial Situation and Rendezvous

The rendezvous between I and T will be attempted by two burn periods (one at the beginning and one at the end of the mission) and a coast phase in between. This is the simplest strategy closer to reality than the two-impulse rendezvous.

No variation of the thrust force is allowed; the engine of I is assumed to be either on or off. But since the mass of I decreases during the burns, the thrust acceleration f increases during the burns with time t according to

$$f(t) = \frac{e}{\tau - t^*} \tag{2.1}$$

where t^* is the current total burn time.

If the first and the second burn durations are denoted by D_0 and D_2 respectively, the accelerations during these burns will be approximated by

$$f_0 = \frac{e}{\tau - \frac{D_0}{2}} \qquad f_2 = \frac{e}{\tau - D_0 - \frac{D_2}{2}} \tag{2.2}$$

respectively. If the total burn duration

$$D = D_0 + D_2 \tag{2.3}$$

is minimized, fuel optimality of the rendezvous (in our approximation) is guaranteed.

Since the optimal trajectory will be sought only among the ones with constant thrust direction during each burn, only discrete angles must be introduced in order to characterize the thrust. We choose the angle χ_0 between the thrust direction and the fixed x_1-direction at $t = 0$ and the angle χ_2 with the same meaning at the rendezvous.

Thus, the quantities D_0, D_2, χ_0, χ_2 are the control parameters to be calculated from the initial data, while quantities s, σ to be introduced later for characterizing the location of the rendezvous are merely unknowns in the mathematical problem.

Section 3
LEVI-CIVITA'S REGULARIZATION

Regularizing is removing singularities from differential equations and their solutions by introducing new variables by appropriate transformations. Methods for doing this depend strongly upon the nature of the singularities that are to be regularized. In the case of the two-body problem in celestial mechanics the corresponding DEQ

$$\frac{d^2 x_j}{dt^2} + \mu \frac{x_j}{r^3} = 0 , \quad j = 1, 2 \tag{3.1}$$

have a singularity at the origin $r = 0$, but in the solution $x_j(t)$ this singularity becomes manifest only when the vehicle collides with the central body.

Regularizations of (3.1) have been known for a long time. In 1906 T. Levi-Civita (Ref. 7) found his regularization of the planar two-body problem, but only recently in 1965 it has been extended to three dimensions by P. Kustaanheimo and E. Stiefel (Refs. 8, 9). The importance of these transformations lies in the fact that they produce not only regular but also linear DEQ for the Keplerian motion.

In the sequel we give a brief outline of Levi-Civita's regularization as well as a collection of the formulas we need for the further development. For the derivations we refer to Ref. 9.

3.1 THE KEPLER MOTION

Levi-Civita's regularization consists of introducing the generalized coordinates u_1, u_2 in the physical x_1, x_2-plane according to the conformal transformation

$$x_1 = u_1^2 - u_2^2 , \quad x_2 = 2u_1 u_2 \tag{3.2}$$

and of introducing the parameter (regularized time)

$$s^* = \int_0^t \frac{d\tau}{r(\tau)} \tag{3.3}$$

as independent variable instead of the time t, where r is the distance of the point (x_1, x_2) from the origin and satisfies the relations

$$r = \sqrt{x_1^2 + x_2^2} = u_1^2 + u_2^2 \tag{3.4}$$

The velocity components $\dot{x}_1$, $\dot{x}_2$ are transformed according to

$$\begin{aligned} \frac{du_1}{ds^*} &= \frac{1}{2} \left(u_1 \dot{x}_1 + u_2 \dot{x}_2 \right) \\ \frac{du_2}{ds^*} &= \frac{1}{2} \left(-u_2 \dot{x}_1 + u_1 \dot{x}_2 \right) \end{aligned} \tag{3.5}$$

The application of the transformation (3.2), (3.3) to the DEQ (3.1) of the unperturbed Kepler motion yields the linear system

$$\frac{d^2 u_j}{ds^{*2}} + \omega_I^2 \, u_j = 0, \quad j = 1, 2 \tag{3.6}$$

where ω_I^2 is an energy constant given for instance by initial values r_I, v_I of distance from the origin and velocity:

$$\omega_I^2 = \frac{1}{2} \left(\frac{\mu}{r_I} - \frac{v_I^2}{2} \right) \tag{3.7}$$

Equations (3.6) are solved by

$$u_j = \alpha_j \cos \omega_I s^* + \beta_j \sin \omega_I s^*$$

$$j = 1, 2 \quad . \qquad (3.8)$$

$$\frac{du_j}{ds^*} = \omega_I (-\alpha_j \sin \omega_I s^* + \beta_j \sin \omega_I s^*)$$

The integration constants α_j, β_j are referred to as the <u>regularized elements</u> of the considered Kepler orbit because they characterize the orbit completely, as the classical elements do.

3.2 PERTURBATIONS

Equations (3.8) give a starting point for handling the perturbed Kepler motion given by the differential equations

$$\frac{d^2 x_j}{dt^2} + \mu \frac{x_j}{r^3} = p_j \qquad (3.9)$$

where p_j are the accelerations due to the perturbing forces. The generalized forces q_j corresponding to the coordinates u_1, u_2 are

$$q_1 = 2(u_1 p_1 + u_2 p_2)$$

$$(3.10)$$

$$q_2' = 2(-u_2 p_1 + u_1 p_2)$$

The presence of perturbing forces causes the frequency ω to be variable (the value ω_I given by (3.7) being merely an initial value), and it generates an inhomogeneous term on the right-hand side of (3.6). Furthermore the elements α_j, β_j defined by (3.8) are now functions of s* rather than constants.

It can be shown (Ref. 9) that by introducing an independent variable, s by a transformation slightly different from (3.3), one can come up with regularized

equations where the frequency ω_I is again constant according to (3.7). The new time transformation involves the semi-major axis a of the ellipse osculating the perturbed Kepler orbit:

$$a = \frac{1}{\frac{2}{r} - \frac{v^2}{\mu}} , \tag{3.11}$$

where v is the interceptor's current velocity. When primes denote differentiation with respect to s, this parameter is defined by the differential equation

$$t' = \sqrt{\frac{a}{a_I}}\, r , \tag{3.12}$$

where

$$a_I = \frac{\mu}{4\omega_I^2} \tag{3.13}$$

is the initial value of the semi-major axis.

Applying the time transformation (3.12) together with the conformal mapping (3.2) onto the differential equation (3.9) of the perturbed Kepler motion yields

$$u_j'' + \omega_I^2\, u_j = F_j , \qquad j = 1, 2 \tag{3.14}$$

with the perturbation functions

$$F_j = \frac{1}{4} \frac{a}{a_I} \left(r\, q_j + \frac{u_j'}{\omega_I^2} \sum_{k=1}^{2} q_k\, u_k' \right) . \tag{3.15}$$

The velocity transformation (3.5) reads now

$$
\begin{aligned}
u_1' &= \frac{1}{2}\sqrt{\frac{a}{a_I}}\,(u_1\,\dot{x}_1 + u_2\,\dot{x}_2) \\
u_2' &= \frac{1}{2}\sqrt{\frac{a}{a_I}}\,(-u_2\,\dot{x}_1 + u_1\,\dot{x}_2) \quad .
\end{aligned}
\tag{3.16}
$$

Equations (3.15) are solved by the method of "varying the constant," which yields equations

$$
\left.
\begin{aligned}
u_j &= \alpha_j(s)\cos\omega_I\, s + \beta_j(s)\sin\omega_I\, s \\
u_j' &= \omega_I\left[-\alpha_j(s)\sin\omega_I\, s + \beta_j(s)\cos\omega_I\, s\right]
\end{aligned}
\right\} \quad j = 1,\ 2
$$

similar to (3.8). With the abbreviation

$$
w_j = \frac{u_j'}{\omega_I} \quad , \qquad j = 1,\ 2 \tag{3.17}
$$

they read in matrix notation

$$
\begin{pmatrix} u_1 & u_2 \\ w_1 & w_2 \end{pmatrix} = \begin{pmatrix} \cos\omega_I s & \sin\omega_I s \\ -\sin\omega_I s & \cos\omega_I s \end{pmatrix} \cdot \begin{pmatrix} \alpha_1(s) & \alpha_2(s) \\ \beta_1(s) & \beta_2(s) \end{pmatrix} \tag{3.18}
$$

The elements $\alpha_j(s)$, $\beta_j(s)$ are now obtained by integrating

$$
\alpha_j' = -\frac{1}{\omega_I}\; F_j \sin\omega_I\, s, \quad \beta_j' = \frac{1}{\omega_I}\; F_j\cos\omega_I\, s \quad . \tag{3.19}
$$

The semi-major axis a can also be written in terms of the elements:

$$
a = \frac{1}{2}\,(\alpha_1{}^2 + \alpha_2{}^2 + \beta_1{}^2 + \beta_2{}^2) \ . \tag{3.20}
$$

Section 4
DERIVATION OF THE CONTROL LAWS

The equations yielding the guidance laws will be established according to the following ideas:

- Describe all motions involved in terms of Levi-Civita's variables and regularized orbital elements.
- When two trajectories match in position and velocity at a certain time, their elements agree.

In the sequel the subscripts 0, 1, 2, 3 (second subscript if there are two) denote values associated with the parking orbit or first burn, coast, second burn or rendezvous, target orbit, respectively.

4.1 MOTION OF THE TARGET

The semi-major axis a_2 and frequency ω_2 associated with the target orbit are given by (3.11) and (3.7):

$$\frac{1}{a_2} = \frac{2}{r_3} - \frac{v_3^2}{\mu} , \quad \omega_2 = \sqrt{\frac{\mu}{4a_2}} \tag{4.1}$$

According to Fig. 1 the target's initial position is defined by the coordinates

$$x_{13} = r_3 \cos \delta , \quad x_{23} = r_3 \sin \delta$$

Inverting Levi-Civita's transformation (3.2) yields the target's first two regularized elements:

$$\alpha_{13} = \sqrt{r_3} \cos \frac{\delta}{2} , \quad \alpha_{23} = \sqrt{r_3} \sin \frac{\delta}{2} \tag{4.2}$$

On the other hand, the initial velocity of T is

$$\dot{x}_{13} = v_3 \sin(\gamma_3 - \delta), \quad \dot{x}_{23} = v_3 \cos(\gamma_3 - \delta) \tag{4.3}$$

as seen in Fig. 1. From the definition (3.17) together with (3.5) we obtain the remaining orbital elements

$$\beta_{13} = \frac{1}{2\omega_2} (\alpha_{13}\dot{x}_{13} + \alpha_{23}\dot{x}_{23})$$

$$\beta_{23} = \frac{1}{2\omega_2} (-\alpha_{23}\dot{x}_{13} + \alpha_{13}\dot{x}_{23})$$

or by using (4.2) and (4.3) ,

$$\begin{aligned} \beta_{13} &= \frac{\sqrt{r_3}\, v_3}{2\omega_2} \sin\left(\gamma_3 - \frac{\delta}{2}\right) \\ \beta_{23} &= \frac{\sqrt{r_3}\, v_3}{2\omega_2} \cos\left(\gamma_3 - \frac{\delta}{2}\right) \end{aligned} \tag{4.4}$$

The quantities α_{j3}, β_{j3} are the initial values for solving the unperturbed regularized system (3.6) defining the target's motion.

We now assign the values $s = 0$ and $s = \sigma$ to the initial point T_3 and to the rendezvous point T_2 on the target orbit, respectively.

According to (3.18) T_2 is then given by the regularized coordinates

$$\begin{pmatrix} u_{12} & u_{22} \\ w_{12} & w_{22} \end{pmatrix} = \begin{pmatrix} \cos\omega_2\sigma & \sin\omega_2\sigma \\ -\sin\omega_2\sigma & \cos\omega_2\sigma \end{pmatrix} \cdot \begin{pmatrix} \alpha_{13} & \alpha_{23} \\ \beta_{13} & \beta_{23} \end{pmatrix} . \tag{4.5}$$

The time t_{tar} taken by the target to move from T_3 to T_2 is calculated from (3.12), putting $\sqrt{a/a_0} = 1$ and using (3.4) and (3.18):

$$t_{tar} = \int_0^{\sigma} \Big[(\alpha_{13} \cos \omega_2 s + \beta_{13} \sin \omega_2 s)^2$$

$$+ (\alpha_{23} \cos \omega_2 s + \beta_{23} \sin \omega_2 s)^2 \Big] \, ds \ ,$$

or

$$t_{tar} = a_2 \sigma + \frac{\alpha_{13} \beta_{13} + \alpha_{23} \beta_{23}}{2\omega_2} (1 - \cos 2\omega_2 \sigma)$$

$$+ \frac{\alpha_{13}^2 + \alpha_{23}^2 - \beta_{13}^2 - \beta_{23}^2}{4\omega_2} \sin 2\omega_2 \sigma \ . \tag{4.6}$$

Here we have used the relation (3.20) which can also be written as

$$a_2 = \frac{1}{2} (u_{12}^2 + u_{22}^2 + w_{12}^2 + w_{22}^2) \tag{4.7}$$

according to (3.18).

4.2 THE BURN PERIODS

In order to deal with the interceptor's motion we must define the quantities

$$\frac{1}{a_0} = \frac{2}{r_0} - \frac{v_0^2}{\mu} \ , \quad \omega_0 = \sqrt{\frac{\mu}{4a_0}} \tag{4.8}$$

analogous to (4.1).

The motion of I will be described in terms of the regularized elements α_j, β_j. Their values α_{j0}, β_{j0} at the beginning of the mission (t = s = 0) agree with the corresponding values u_{j0}, w_{j0} of the regularized coordinates. These are found by a sequence of transformations similar to (4.2) through (4.4). Taking into account the particular situation of Fig. 1, we obtain

$$u_{10} = \alpha_{10} = \sqrt{r_0}\,, \qquad u_{20} = \alpha_{20} = 0$$

$$w_{10} = \beta_{10} = \frac{v_0}{2}\,\frac{\sqrt{r_0}}{\omega_0}\,\sin\gamma_0, \quad w_{20} = \beta_{20} = \frac{v_0}{2}\,\frac{\sqrt{r_0}}{\omega_0}\,\cos\gamma_0 \tag{4.9}$$

The <u>intermediate elements</u> (the constant values of the elements during the coast) will be denoted by α_{j1}, β_{j1}, and the final elements α_{j2}, β_{j2} (at the rendezvous T_2) are related to the regularized coordinates u_{j2}, w_{j2} by means of (3.18):

$$\begin{pmatrix} u_{12} & u_{22} \\ w_{12} & w_{22} \end{pmatrix} = \begin{pmatrix} \cos\omega_0 s & \sin\omega_0 s \\ -\sin\omega_0 s & \cos\omega_0 s \end{pmatrix} \cdot \begin{pmatrix} \alpha_{12} & \alpha_{22} \\ \beta_{12} & \beta_{22} \end{pmatrix}. \tag{4.10}$$

The unknown quantity s now stands for the value of the regularized time on the interceptor's orbit at T_2.

We further introduce the element increments

$$\begin{aligned} \Delta\alpha_j &= \alpha_{j2} - \alpha_{j0} \\ \Delta\beta_j &= \beta_{j2} - \beta_{j0} \end{aligned} \qquad j = 1,\ 2 \tag{4.11}$$

which are caused by the perturbing effect of the thrust during the burn periods.

The next step is to relate the element increments to the control parameters. These relations are based upon an approximate solution of the DEQ (3.19) for the

burns. The right-hand sides will be evaluated at the rendezvous during the entire second burn (at the beginning $t = s = 0$ during the first burn). This is the principle of first order perturbations. Thus we obtain

$$\begin{aligned}
\alpha_{j1} - \alpha_{j0} &= 0 \\
\beta_{j1} - \beta_{j0} &= \frac{s_0}{\omega_0} F_{j0} \\
\alpha_{j2} - \alpha_{j1} &= - \frac{s_2}{\omega_0} F_{j2} \sin \omega_0 s \qquad j = 1,\ 2 \\
\beta_{j2} - \beta_{j1} &= \frac{s_2}{\omega_0} F_{j2} \cos \omega_0 s
\end{aligned} \tag{4.12}$$

where s_0 and s_2 are the increments of the regularized time in the first and second burn, respectively, and F_{j0} and F_{j2} are the values of F_j at the beginning and at the rendezvous, respectively.

From (4.11) and (4.12) there follows

$$\begin{aligned}
\Delta\alpha_j &= - \frac{s_2}{\omega_2} F_{j2} \sin \omega_0 s \\
\Delta\beta_j &= \frac{s_2}{\omega_0} F_{j2} \cos \omega_0 s + \frac{s_0}{\omega_0} F_{j0}
\end{aligned} \tag{4.13}$$

and, by eliminating F_{j2} from these two equations we obtain

$$\Delta\gamma_j = \frac{s_0}{\omega_0} F_{j0} \sin \omega_0 s , \tag{4.14}$$

where

$$\Delta\gamma_j = \Delta\alpha_j \cdot \cos \omega_0 s + \Delta\beta_j \sin \omega_0 s \tag{4.15}$$

Equations (4.13) and (4.14) will further be used; to this end we prepare the expression for F_{j2} from (3.16):

$$F_{j2} = \frac{1}{4}\frac{\omega_0^2}{\omega_2^2}\left[r_2 q_j + \frac{u_j'}{\omega_0^2}(q_1 u_1' + q_2 u_2')\right]$$
$$= \frac{1}{4}\frac{\omega_0^2}{\omega_2^2} \cdot \sum_{k=1}^{2} (r\,\delta_{jk} + w_{j2} w_{k2})\, q_{k2} \,. \tag{4.16}$$

Here δ_{jk} is the Kronecker symbol, and r_2, q_{j2} are given by

$$r_2 = u_{12}^2 + u_{22}^2 \,, \tag{4.17}$$

$$\begin{pmatrix} q_{12} \\ q_{22} \end{pmatrix} = \frac{2e}{\tau - D_0 - \frac{D_2}{2}} \begin{pmatrix} u_{12} & u_{22} \\ -u_{22} & u_{12} \end{pmatrix} \begin{pmatrix} \cos X_2 \\ \sin X_2 \end{pmatrix} \tag{4.18}$$

according to (3.4), (3.10), (2.2). The quantities u_{j2}, w_{j2} are defined in (4.5). Thus, (4.13) is of the form

$$\begin{pmatrix} \Delta\alpha_1 \\ \Delta\alpha_2 \end{pmatrix} = -M \begin{pmatrix} \cos X_2 \\ \sin X_2 \end{pmatrix} \tag{4.19}$$

where M is the matrix

$$M = \frac{s_2}{2}\;\frac{e \cdot \sin\omega_0 s}{\tau - D_0 - \frac{D_2}{2}}\;\frac{\omega_0}{\omega_2^2} \cdot \begin{pmatrix} r_2 + w_{12}^2 & w_{12} w_{22} \\ w_{12} w_{22} & r_2 + w_{22}^2 \end{pmatrix} \begin{pmatrix} u_{12} & u_{22} \\ -u_{22} & u_{12} \end{pmatrix} \tag{4.20}$$

The two equations (4.19) easily allow the elimination of the unknown thrust direction X_2 (we intend to keep only the unknowns s and σ in the equations):

$$(\Delta\alpha_1 \quad \Delta\alpha_2)\ (MM^T)^{-1} \begin{pmatrix} \Delta\alpha_1 \\ \Delta\alpha_2 \end{pmatrix} = 1 \quad , \tag{4.21}$$

where M^T is the transpose of M. We record a few intermediate results of the evaluation of this matrix product: From (4.20) we obtain

$$MM^T = r_2 \left(\frac{s_2}{2} \frac{\omega_0}{\omega_2^{\ 2}} \cdot \frac{e}{\tau - D_0 - \frac{D_2}{2}} \sin \omega_0 s \right)^2 \cdot$$

$$\begin{pmatrix} r_2 + w_{12} & w_{12}\, w_{22} \\ w_{12}\, w_{22} & r_2 + w_{22}^{\ 2} \end{pmatrix}^2$$

by using (4.17), and the inversion yields

$$(MM^T)^{-1} = \left(s_2\, a_2\, r_2^{\ 3/2} \frac{\omega_0}{\omega_2^{\ 2}} \cdot \frac{e}{\tau - D_0 - \frac{D_2}{2}} \sin \omega_0 s\right)^2 \cdot$$

$$\cdot \begin{pmatrix} r_2 + w_{22}^{\ 2} & -w_{12}\, w_{22} \\ -w_{12}\, w_{22} & r_{12} + w_{12}^{\ 2} \end{pmatrix}^2 \tag{4.22}$$

when (4:7) is applied. Multiplying the matrix in (4.22) from the left and from the right by the vector $(\Delta\alpha_1, \ \Delta\dot{\alpha}_2)$ further yields the expression

$$r_2^{\ 2} (\Delta\alpha_1^{\ 2} + \Delta\alpha_2^{\ 2}) + (r_2 + a_2)(w_{22}\, \Delta\alpha_1 - w_{12}\, \Delta\alpha_2)^2 \ , \tag{4.23}$$

when again (4.7) is used.

Finally, the time equation (3.12) is integrated approximately (first order perturbations) resulting in

$$D_0 = r_0 s_0, \quad D_2 = \frac{\omega_0}{\omega_2} r_2 s_2 \tag{4.24}$$

for the burn durations D_0, D_2. Using equations (4.22) through (4.24) in (4.21) now yields the equation

$$\frac{\mu e}{4} \sin \omega_0 s \cdot D_2 = G_2 \left(\tau - D_0 - \frac{D_2}{2}\right) \tag{4.25}$$

with

$$G_2 = \omega_2^{\,3} \sqrt{r_2(\Delta\alpha_1^{\,2} + \Delta\alpha_2^{\,2}) + \left(1 + \frac{a_2}{r_2}\right)(w_{22}\,\Delta\alpha_1 - w_{12}\,\Delta\alpha_2)^2} \tag{4.26}$$

which contains no other unknowns than s, σ, D_0, D_2 .

A similar equation can be derived from (4.14) in the same way; the result is

$$\frac{\mu e}{4} \sin \omega_0 s \cdot D_0 = G_0 \left(\tau - \frac{D_0}{2}\right) \tag{4.27}$$

with

$$G_0 = \omega_0^{\,3} \sqrt{r_0 (\Delta\gamma_1^{\,2} + \Delta\gamma_2^{\,2}) + \left(1 + \frac{a_0}{r_0}\right)(w_{20}\,\Delta\gamma_1 - w_{10}\,\Delta\gamma_2)^2} \tag{4.28}$$

and $\Delta\gamma_j$ from (4.15).

In order to obtain the total burn duration D we first solve (4.27) for D_0:

$$D_0 = \frac{\tau\, G_0}{\lambda + \frac{1}{2} G_0} \tag{4.29}$$

where λ is the abbreviation

$$\lambda = \frac{1}{4} \mu \text{ e } \sin \omega_0 \text{ s} \tag{4.30}$$

Using this in (4.25) then yields

$$D_2 = \frac{\tau \, G_2 \, (\lambda - \frac{1}{2} \, G_0)}{(\lambda + \frac{1}{2} G_0) \, (\lambda + \frac{1}{2} G_2)} \tag{4.31}$$

Hence the total burn duration $D = D_0 + D_2$ is the function

$$D(s, \sigma) = \frac{\tau \lambda (G_0 + G_2)}{(\lambda + \frac{1}{2} G_0) \, (\lambda + \frac{1}{2} G_2)} = \min \tag{4.32}$$

which is to minimize according to the requirement of fuel optimality by appropriate choice of s and σ .

4.3 THE COAST

The coast trajectory of the interceptor is characterized by the intermediate elements α_{j1}, β_{j1} which can be obtained from the first and the last two equations of (4.12)

$$\begin{aligned} \alpha_{j1} &= \alpha_{j0} \\ \beta_{j1} &= \beta_{j2} + \Delta\alpha_j \cdot \operatorname{cotan} \omega_0 \, s \end{aligned} \qquad j = 1, 2 \tag{4.33}$$

These quantities allow us to define the coast semi-major axis a_1 and the corresponding frequency ω_1:

$$a_1 = \frac{1}{2} (\alpha_{11}^2 + \alpha_{21}^2 + \beta_{11}^2 + \beta_{21}^2)$$
$$\omega_1 = \sqrt{\frac{\mu}{4a_1}} \tag{4.34}$$

Now we can calculate the time t_{coa} the interceptor takes for the coast by integrating (3.12) from s_0 to $s - s_2$:

$$t_{coa} = \sqrt{\frac{a_1}{a_0}} \cdot \int_{s_0}^{s-s_2} \Big[(\alpha_{11} \cos \omega_1 s + \beta_{11} \sin \omega_1 s)^2$$
$$+ (\alpha_{21} \cos \omega_1 s + \beta_{21} \sin \omega_1 s)^2 \Big] ds$$

or

$$t_{coa} = \sqrt{\frac{a_1}{a_0}} \Big[a_1 (s - s_0 - s_2)$$
$$+ \frac{\alpha_{11} \beta_{11} + \alpha_{21} \beta_{21}}{2\omega_1} (\cos 2\omega_1 s_0 - \cos 2\omega_1 (s - s_2)) \tag{4.35}$$
$$+ \frac{\alpha_{11}^2 + \alpha_{21}^2 - \beta_{11}^2 - \beta_{21}^2}{4\omega_1} (\sin 2\omega_1 (s - s_2) - \sin 2\omega_1 s_0) \Big],$$

expressions for s_0, s_2 in terms of s, σ are obtained from (4.24):

$$s_0 = \frac{D_0}{r_0} , \qquad s_2 = \frac{\omega_2}{\omega_0} \frac{D_2}{r_2} \tag{4.36}$$

The condition

$$Z (s, \sigma) = t_{coa} + D - t_{tar} = 0 \tag{4.37}$$

consequently guarantees that the two vehicles arrive at the rendezvous location simultaneously.

Thus the rendezvous problem is reduced to the problem of minimizing the function $D(s, \sigma)$ while the side condition $Z(s, \sigma) = 0$ must be satisfied.

If once s and σ are calculated, the burn durations are found from (4.29) and (4.31). Starting with (4.19) we will finally establish equations for the thrust angles X_0, X_2. Equation (4.19) can be written as

$$\begin{pmatrix} \Delta\alpha_1 \\ \Delta\alpha_2 \end{pmatrix} = \text{const} \begin{pmatrix} r_2 + w_{12}^2 & w_{12} w_{22} \\ w_{12} w_{22} & r_2 + w_{22}^2 \end{pmatrix} \begin{pmatrix} \cos (X_2 - \varphi_2) \\ \sin (X_2 - \varphi_2) \end{pmatrix} \tag{4.38}$$

where

$$\varphi_2 = \arg (u_{12} + i\, u_{22}). \tag{4.39}$$

Inverting (4.38) and forming the quotient yields

$$\tan (X_2 - \varphi_2) = \frac{r_2 \Delta\alpha_2 - w_{12} (w_{22} \Delta\alpha_1 - w_{12} \Delta\alpha_2)}{r_2 \Delta\alpha_1 + w_{22} (w_{22} \Delta\alpha_1 - w_{12} \Delta\alpha_2)} \tag{4.40}$$

Starting from (4.14) we similarly obtain (since $\varphi_0 = 0$)

$$\tan X_0 = \frac{r_0 \Delta\gamma_2 - w_{10} (w_{20} \Delta\gamma_1 - w_{10} \Delta\gamma_2)}{r_0 \Delta\gamma_1 + w_{20} (w_{20} \Delta\gamma_1 - w_{10} \Delta\gamma_2)} \tag{4.41}$$

Section 5
THE CONTROL LAWS

Here we summarize the equations derived in the last section in an order appropriate for computer programming. The numbers at the right-hand side refer to the corresponding equations in the previous sections.

Input Variables

$\mu, \tau, e, \delta, r_0, v_0, \gamma_0, r_3, v_3, \gamma_3$

Constants (Independent of the main unknowns s, σ)

$$a_0 = \left(\frac{2}{r_0} - \frac{v_0^2}{\mu} \right)^{-1} , \quad \omega_0 = \sqrt{\mu/4a_0} \tag{4.8}$$

$$a_2 = \left(\frac{2}{r_3} - \frac{v_3^2}{\mu} \right)^{-1} , \quad \omega_2 = \sqrt{\mu/4a_2} \tag{4.1}$$

$$\alpha_{10} = \sqrt{r_0} , \quad \alpha_{20} = 0 \tag{4.9}$$

$$\beta_{10} = \frac{\sqrt{r_0}}{\omega_0} \frac{v_0}{2} \sin \gamma_0 , \quad \beta_{20} = \frac{\sqrt{r_0}}{\omega_0} \frac{v_0}{2} \cos \gamma_0 \tag{4.9}$$

$$\alpha_{13} = \sqrt{r_3} \cos \frac{\delta}{2} , \quad \alpha_{23} = \sqrt{r_3} \sin \frac{\delta}{2} \tag{4.2}$$

$$\beta_{13} = \frac{\sqrt{r_3}}{\omega_2} \cdot \frac{v_3}{2} \sin \left(\gamma_3 - \frac{\delta}{2} \right), \quad \beta_{23} = \frac{\sqrt{r_3}}{\omega_2} \cdot \frac{v_3}{2} \cos \left(\gamma_3 - \frac{\delta}{2} \right) \tag{4.4}$$

The Functions $D(s,\sigma)$, $Z(s,\sigma)$

$$\begin{pmatrix} u_{12} & u_{22} \\ w_{12} & w_{22} \end{pmatrix} = \begin{pmatrix} \cos\omega_2\sigma & \sin\omega_2\sigma \\ -\sin\omega_2\sigma & \cos\omega_2\sigma \end{pmatrix} \begin{pmatrix} \alpha_{13} & \alpha_{23} \\ \beta_{13} & \beta_{23} \end{pmatrix} \tag{4.5}$$

$$\begin{pmatrix} \alpha_{12} & \alpha_{22} \\ \beta_{12} & \beta_{22} \end{pmatrix} = \begin{pmatrix} \cos\omega_0 s & -\sin\omega_0 s \\ \sin\omega_0 s & \cos\omega_0 s \end{pmatrix} \begin{pmatrix} u_{12} & u_{22} \\ w_{12} & w_{22} \end{pmatrix} \tag{4.10}$$

$$r_2 = u_{12}^2 + u_{22}^2 \tag{4.17}$$

$$\Delta\alpha_j = \alpha_{j2} - \alpha_{j0} \qquad j = 1,\ 2 \tag{4.11}$$

$$\Delta\gamma_j = \Delta\alpha_j \cos\omega_0 s + (\beta_{j2} - \beta_{j0}) \sin\omega_0 s \tag{4.15}$$

$$G_0 = \omega_0^3 \sqrt{r_0 (\Delta\gamma_1^2 + \Delta\gamma_2^2) + (1 + \frac{a_0}{r_0})(\beta_{20}\ \Delta\gamma_1 - \beta_{10}\ \Delta\gamma_2)^2} \tag{4.28}$$

$$G_2 = \omega_2^3 \sqrt{r_2 (\Delta\alpha_1^2 + \Delta\alpha_2^2) + (1 + \frac{a_2}{r_2})(w_{22}\,\Delta\alpha_1 - w_{12}\,\Delta\alpha_2)^2} \tag{4.26}$$

$$\lambda = \frac{1}{4}\mu e \sin\omega_0 s \tag{4.30}$$

$$D_0 = \frac{\tau G_0}{\lambda + \frac{1}{2} G_0} \tag{4.29}$$

$$D(s, \sigma) = D = \frac{\tau\lambda(G_0 + G_2)}{(\lambda + \frac{1}{2} G_0)(\lambda + \frac{1}{2} G_2)} \tag{4.32}$$

$$D_2 = D - D_0 \tag{2.3}$$

$$\beta_{j1} = \beta_{j2} + \Delta\alpha_j \cot \omega_0 s \qquad j = 1, 2 \tag{4.33}$$

$$a_1 = \frac{1}{2}(r_0 + \beta_{11}{}^2 + \beta_{21}{}^2), \quad \omega_1 = \sqrt{\mu/4a_1} \tag{4.34}$$

$$s_0 = \frac{D_0}{r_0}, \qquad s_2 = \frac{\omega_2}{\omega_0} \frac{D_2}{r_2} \tag{4.36}$$

$$\begin{aligned} t_{coa} = \frac{\omega_0}{\omega_1} \Big[& a_1 (s - s_0 - s_2) \\ & + \frac{1}{2\omega_1} (\alpha_{10}\beta_{11} + \alpha_{20}\beta_{21})(\cos 2\omega_1 s_0 - \cos 2\omega_1 (s - s_2)) \\ & + \frac{1}{2\omega_1} (r_0 - a_1)(\sin 2\omega_1 (s - s_2) - \sin 2\omega_1 s_0) \Big] \end{aligned} \tag{4.35}$$

$$\begin{aligned} t_{tar} = {} & a_2\sigma + \frac{1}{2\omega_2}(\alpha_{13}\beta_{13} + \alpha_{23}\beta_{23})(1 - \cos 2\omega_2 \sigma) \\ & + \frac{1}{2\omega_2}(r_3 - a_2)\sin 2\omega_2 \sigma \end{aligned} \tag{4.6}$$

$$Z(s, \sigma) = Z = t_{coa} + D - t_{tar} \tag{4.37}$$

The minimum problem D = minimum with the side condition Z = 0 can be solved iteratively without using derivatives of the function $D(s, \sigma)$ for instance by the method described at the end of this section.

Output

The variables s, σ , s_0, s_2, D_0, D_2, D, t_{tar} are available from the computation of the functions $D(s,\sigma)$, $Z(s,\sigma)$. In addition χ_0 and χ_2 are obtained from

$$\tan \chi_0 = \frac{r_0 \, \Delta\gamma_2 - \beta_{10} \, (\beta_{20} \, \Delta\gamma_1 - \beta_{10} \, \Delta\gamma_2)}{r_0 \, \Delta\gamma_1 + \beta_{20} \, (\beta_{20} \, \Delta\gamma_1 - \beta_{10} \, \Delta\gamma_1)} \tag{4.41}$$

$$\varphi_2 = \arg \, (u_{12} + i \, u_{22}) \tag{4.39}$$

$$\tan \, (\chi_2 - \varphi_2) = \frac{r_2 \, \Delta\alpha_2 - w_{12} \, (w_{22} \, \Delta\alpha_1 - w_{12} \, \Delta\alpha_2)}{r_2 \, \Delta\alpha_1 + w_{22} \, (w_{22} \, \Delta\alpha_1 - w_{12} \, \Delta\alpha_2)} \tag{4.40}$$

The Minimization Technique

The idea is to seek the smallest value of the function D along the line $Z(s,\sigma) = 0$ of the (s,σ)-plane. A two-stage iteration process is used to improve an appropriate initial guess s_0, σ_0 almost to machine accuracy. A step size h indicating the order of magnitude of the error in the initial guess must be known.

In the first stage for the three fixed abscissas s_0, $s_1 = s_0 + h$, $s_2 = s_0 - h$ the one-dimensional secant method (starting from σ_0) is used to find values $\sigma = \sigma_j^*$ (j=0, 1, 2) which approximately zero the function $Z(s,\sigma)$ at $s = s_j$:

$$\max_j \left| Z(s_j, \, \sigma_j^*) \right| \leq \epsilon_1 \left| Z(s_0, \, \sigma_0) \right| , \tag{5.1}$$

where ϵ_1 is a small positive number, for instance $\epsilon_1 = 0.005$.

A necessary condition for the feasibility of these operations is

$$\frac{\partial Z}{\partial \sigma} \neq 0 \tag{5.2}$$

in the region in which the arguments vary during the process. If (5.2) is violated the method works when the roles of s and σ are exchanged.

In the second stage the values $D_j = D(s_j, \sigma_j^*)$ are calculated. Quadratic interpolation then yields the approximate abscissa s_m of an extreme value of $D(s, \sigma)$ with $Z(s, \sigma) = 0$:

$$s_m = s_0 - \frac{h}{2} \quad \frac{D_1 - D_2}{D_1 - 2D_0 + D_2} \tag{5.3}$$

Finally, the corresponding value σ_m is calculated by quadratic interpolation with the collocation points s_j and the values σ_j^*:

$$\sigma_m = \sigma_0^* + (s_m - s_0) \frac{\sigma_1^* - \sigma_2^*}{2h} + (s_m - s_0)^2 \frac{\sigma_1^* - 2\sigma_0^* + \sigma_2^*}{2h^2} . \tag{5.4}$$

The iteration cycle is closed by assigning the values s_m, σ_m to the variables s_0, σ_0 and by taking

$$h = \epsilon_2 (s_m - s_0)$$

as the new step size, where ϵ_2 is another small number, for example $\epsilon_2 = 0.1$. Figure 2 illustrates the meaning of the various quantities introduced here.

Applied to the case of the functions $D(s, \sigma)$, $Z(s, \sigma)$ this technique converges very fast.

Tabulation of D and Z for typical rendezvous situations has shown that the solution of the minimum problem is unique in the rectangle

$$0 \leq \omega_0 s \leq \frac{\pi}{2} , \qquad 0 \leq \omega_2 \sigma \leq \frac{\pi}{2}$$

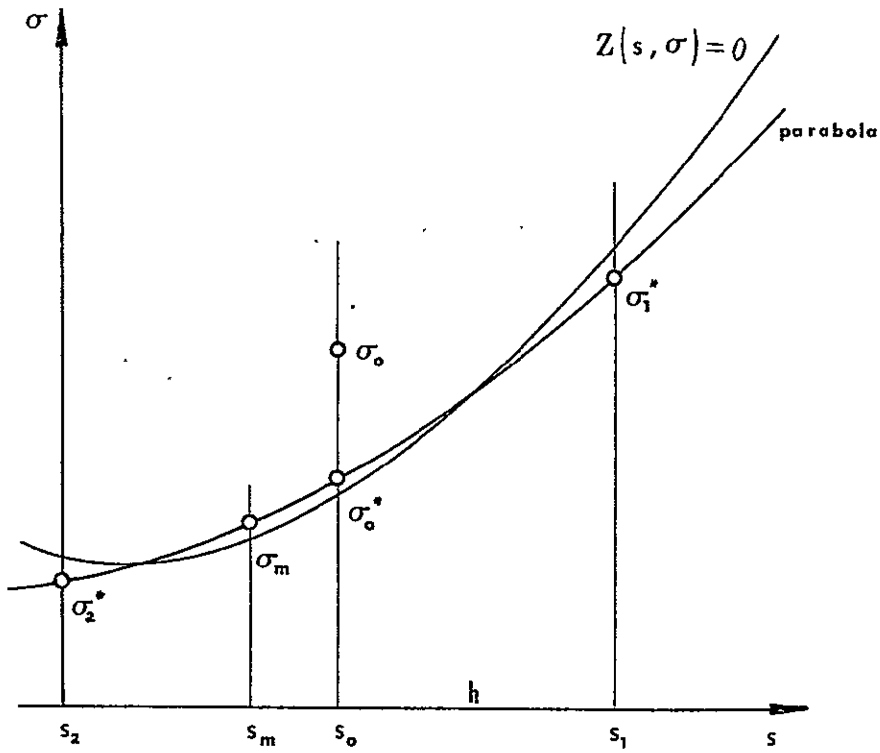

Fig. 2 - Minimization Technique

Furthermore the line Z = 0 lies inside a narrow strip around the line $\omega_0 s = \omega_2 \sigma$, and the partial derivatives of Z in this strip are quite large. Thus, the line Z = 0 is well defined.

With the initial guesses

$$s_0 = \frac{\pi}{2\omega_0} \qquad \sigma_0 = \frac{\pi}{2\omega_2}$$

convergence was achieved in all cases considered. Results of a five digits' accuracy were obtained with less than 25 evaluations of the functions D and Z.

A Fortran IV subprogram FCT(S, SIG) for the evaluation of the functions $D(s, \sigma)$ and $Z(s, \sigma)$ consists of about 40 statements. A calling program (the program CONTRL) which determines the control parameters from the current position and velocity data, can be written with some 80 statements. The run time of the program CONTRL on an IBM7094 computer is in the order of 0.1 sec.

Section 6
OTHER PARAMETER SYSTEMS

The set of the control equations recorded in Section 5 is not too complicated. However, many simplifying assumptions have been used. The most serious one is the linear approximation during the burn periods. The solution $\vec{y}_1 = \vec{y}(h)$ of the system of DEQ

$$\frac{d\vec{y}}{ds} = \vec{g}\,(\vec{y},\ s)\ , \qquad \vec{y}(0) = \vec{y}_0 \tag{6.1}$$

was approximated by

$$\vec{y}_1 = \vec{y}_0 + h\,\vec{g}\,(\vec{y}_0,\ 0)\ . \tag{6.2}$$

A much better approximation would be the value obtained from applying the trapezoidal rule to the DEQ (6.1)

$$\vec{y}_1 = \vec{y}_0 + \frac{h}{2}\left[\vec{g}\,(\vec{y}_0,\ 0) + \vec{g}\,(\vec{y}_1,\ h)\right]. \tag{6.3}$$

Here one would introduce two more unknown control parametets, namely the thrust directions at the beginning and at the end of the coast. This would weaken the assumptions of constant thrust directions, but one would have to solve a minimum problem in four variables.

This idea might be applied in connection with the third parameter system mentioned in Section 1. There is a good chance that the simplicity of the equations corresponding to these parameters compensates somewhat for the complication of introducing two more unknowns. A summary of the most important relations and properties associated with these parameters is given here. For more details see Ref. 6.

We start with the DEQ (3.9) of the perturbed Kepler motion and introduce the direction unit vector

$$y_j = \frac{x_j}{r} \quad , \quad j = 1, 2 \tag{6.4}$$

and the reciprocal distance

$$\rho = \frac{1}{r} \quad . \tag{6.5}$$

These quantities define the position of the vehicle uniquely. If the parameter φ is introduced by

$$dt = r^2 \, d\varphi \, , \qquad t = \int r^2 \, d\varphi \, , \qquad ' \equiv \frac{d}{d\varphi} \tag{6.6}$$

the dependent variables y_j, ρ and t satisfy the DEQ

$$\begin{aligned} y_j'' + \mu \ell y_j &= \frac{1}{\rho^3} \left[p_j - y_j \sum_1^2 p_k y_k \right] , \quad j = 1, 2 \\ \rho'' + \mu \ell \rho &= \mu - \frac{1}{\rho^2} \sum_1^2 p_k y_k \end{aligned} \tag{6.7}$$

$$t' = \frac{1}{\rho^2} \tag{6.8}$$

where ℓ is the semi-latus rectum of the osculating Kepler orbit. ℓ is defined by

$$\ell = \frac{1}{\mu} \cdot \left(x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} \right)^2$$

and satisfies the DEQ

$$\ell' = \frac{2}{\mu \rho^3} \cdot \sum_1^2 p_k y_k' \quad . \tag{6.9}$$

If there are no perturbations ($p_k = 0$) Eqs. (6.7) are linear DEQ in y_j, ρ with constant coefficients (because of (6.9)). They describe a harmonic oscillator with the center $y_j = 0$, $\rho = \frac{1}{\ell}$. Hence, a simple treatment of first order perturbations is possible, which is analogous to the method applied to Levi-Civita's variables in Section 4. In addition the parameters y_j, ρ and the corresponding orbital elements (Ref. 6) do not show any singular behavior in a transition through a circular orbit.

In the derivation of the rendezvous conditions one can take advantage of the simple geometric relations given by (6.4) (6.5) and of the fact that in the unperturbed case the independent variable φ is proportional to the vehicle's true anomaly.

Section 7
COMPUTER SIMULATION

Simulating a rendezvous consists of imitating on a computer all operations influencing the trajectories of the two vehicles. The motion on their orbits is represented by theoretical or numerical solutions of the corresponding differential equations.

For handling the coast phases it is necessary to calculate the position and velocity vector of the interceptor where it is influenced by the earth's gravitation only. This is the initial value problem of the Keplerian motion. Levi-Civita's variables allow solution to this problem in a stable and efficient way.

From the vehicle's initial coordinates x_1, x_2 and initial velocity components $\dot{x}_1$, $\dot{x}_2$ (at time t=0, relative to an inertial coordinate system centered at the earth's center) the corresponding regularized coordinates and elements can be calculated by formulas of Sections 3 and 4:

$$r = \sqrt{x_1^2 + x_2^2}$$

$$a = \frac{1}{\frac{2}{r} - \frac{\dot{x}_1^2 + \dot{x}_2^2}{\mu}} \qquad (3.11)$$

$$\omega = \sqrt{\mu/4a} \qquad (3.7)$$

(μ is the earth's gravitational parameter)

$$\alpha_1 = \sqrt{\tfrac{1}{2}(r + x_1)} \ , \qquad \alpha_2 = \frac{x_2}{2\alpha_1}$$

$$\beta_1 = \frac{1}{2\omega}\,(\alpha_1\dot{x}_1 + \alpha_2\dot{x}_2), \qquad \beta_2 = \frac{1}{2\omega}\,(-\alpha_2\dot{x}_1 + \alpha_1\dot{x}_2) \tag{3.5}$$

These quantities being known, it is possible to establish the equation

$$\Delta t = a\sigma + \frac{\sin\omega\sigma}{\omega} \cdot \left[(r-a)\cos\omega\sigma + (\alpha_1\beta_1 + \alpha_2\beta_2)\sin\omega\sigma\right] \tag{4.6}$$

which relates the true increment Δt with the parameter value σ corresponding to the vehicle's position at time $t = \Delta t$. This is essentially Kepler's equation; it is most efficiently solved for σ by Newton-Raphson's iteration starting with the initial approximation

$$\sigma_0 = \frac{1}{a}\left(\Delta t - \frac{\alpha_1\beta_1 + \alpha_2\beta_2}{2\omega}\right)$$

Finally, the transformations

$$\begin{pmatrix} u_1 & u_2 \\ w_1 & w_2 \end{pmatrix} = \begin{pmatrix} \cos\omega\sigma & \sin\omega\sigma \\ -\sin\omega\sigma & \cos\omega\sigma \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \tag{4.5}$$

$$x_1 = u_1^2 - u_2^2 \ , \qquad x_2 = 2u_1 u_2 \tag{3.2}$$

$$\dot{x}_1 = \frac{2\omega}{u_1^2 + u_2^2}(u_1 w_1 - u_2 w_2)$$

$$\dot{x}_2 = \frac{2\omega}{u_1^2 + u_2^2}(u_2 w_1 + u_1 w_2)$$

yield quantities x_1, x_2, $\dot{x}_1$, $\dot{x}_2$ representing now the position and velocity of the vehicle at time $t = \Delta t$. A subprogram KEPLER collects the set of the above formulas.

The trajectory of the interceptor during the burn periods is calculated by numerical integration (for example Runge-Kutta) of the differential equations

$$\begin{aligned} \ddot{x}_1 &= -\frac{\mu}{r^3} x_1 + \frac{e}{\tau - t^*} \cos X \\ \ddot{x}_2 &= -\frac{\mu}{r^3} x_2 + \frac{e}{\tau - t^*} \sin X \end{aligned} \tag{7.1}$$

in the Cartesian coordinates x_1, x_2, where X is the current thrust direction measured from the x_1-axis.

The two computer programs CONTRL and KEPLER as well as the subprogram INTGRT for the Runge-Kutta integration of (7.1) are put together to a simulation program according to the rough flow chart shown in Fig. 3.

The control parameters are updated during the first burn only, when the logical variable UPD is TRUE. The fixed updating time interval is DT, while TINT denotes the current updating interval.

The computer output sheets (Fig. 3) give an account of the two vehicles' motions and of the control parameters for two test cases. The lines in the

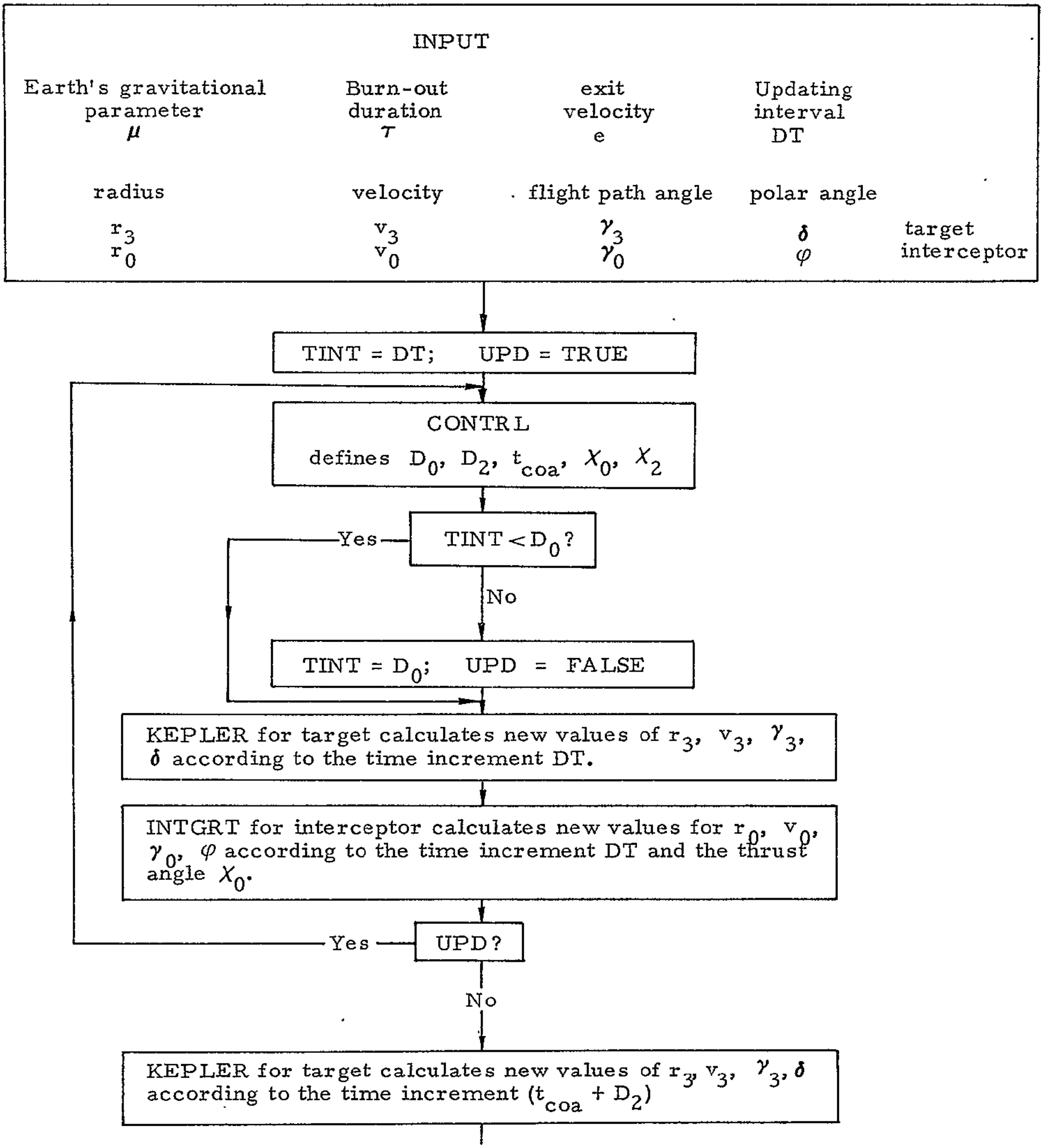

Fig. 3 - Flow Chart of the Simulation Program

OUTPUT

r_3, v_3, γ_3, δ final target data

KEPLER for inteceptor calculates new values of r_0, v_0, γ_0, φ according to the time increment t_{coa}.

INTGRT for interceptor calculates new values of r_0, v_0, γ_0, φ according to the time increment D_2 and the thrust angle χ_2.

OUTPUT

r_0, v_0, γ_0, φ final interceptor data

Fig. 3 - Flow Chart of the Simulation Program (Continued)

section Control correspond to successive iterations in the minimization procedure. The last line gives the values of the control parameters.

As an example, the circular rendezvous case mentioned in Ref. 2, page 9 is considered. Initially, the interceptor and target are on circular orbits 100 km and 400 km above the earth's surface, the target being 5° ahead of the interceptor. The table in Fig. 4 compares the results of the present simulation program with the exact calculus-of-variations solution (COV) and with the results of the guidance scheme in Ref. 2. In the present scheme three cases are considered: (1) no updating (DT $=\infty$), (2) updating interval DT = 10 sec, (3) DT = 4 sec.

	COV	Present Scheme			Ref. 2
		DT= ∞	DT=10	DT=4	
First burn D_0 (sec)	13.29	13.20	13.29	13.31	13.3
Coast t_{coa} (sec)	2522.7	2505.2	2512.2	2508.9	2318.4
Second burn D_2 (sec)	11.97	11.76	11.96	12.00	12.1
Thrust direction χ_0 (deg)	64.9°	64.6°	64.6°	64.6°	62.8°
Thrust direction at end of First Burn χ_1 (deg)	66.1°	64.6°	66.3°	67.0°	
Final thrust direction χ_2 (deg)	-129.4°	-106.0°	-106.5°	-105.0°	-113.8°
Position error (m)	0	5616	971	566	
Velocity error (m/sec)	0	98	100	100	

Fig. 4 - Comparison of Simulation Results

In the second example, the results of a rendezvous with a target moving on an elliptic orbit with semi-major axis 6793 km and eccentricity 0.0100 are shown. The interceptor is initially 8.6° behind the target and 6400 km away from the earth's center. Its orbit has the eccentricity 0.0246. Initially, the first burn, the coast and the second burn are predicted to last for 35.0 sec,

2600.7 sec and 15.2 sec, respectively. Updating throughout the first burn in intervals DT = 10 sec modified these numbers to 35.8 sec, 2628.3 sec, 15.9 sec. The final position and velocity errors of 9317 m and 137 m/sec, respectively, are in the same order of magnitude as the errors in the circular case. The results in this case are collected in Fig. 5.

Further testing showed that the present flight scheduling and guidance scheme yields good results when the assumptions of the scheme — small eccentricity in all Keplerian ellipses and short burn durations — are satisfied. A closed-loop guidance scheme being applied to the second burn, however, would necessarily use a burn duration somewhat longer than the predicted one in order to compensate for the more pronounced velocity errors.

	COV	Present Scheme		
		DT= ∞	DT=20	DT=10
First burn D_0(sec)		34.96	35.65	35.82
Coast t_{coa} (sec)		2600.7	2626.2	2628.3
Second burn D_2 (sec)		15.16	15.64	15.90
Thrust direction X_0 (deg)		129.8°	129.8°	129.8°
Thrust direction at end of First Burn X_1 (deg)		129.8°	132.0°	135.8°
Final thrust direction X_2 (deg)		-83.5°	-81.6°	-80.4°
Position error (m)	0	37653	29842	9317
Velocity error (m/sec)	0	144	145	137

Fig. 5 - Elliptic Case

CIRCULAR CASE, NO UPDATING

INPUT

MU	TAU	E	DT	0.398335E 15	0.583320E 03	0.418740E 04	0.990000E 02
R3	V3	GA3	DEL	0.677330E 07	0.760974E 04	0.000000E 00	0.872665E-1
R0	V0	GA0	PHI	0.647330E 07	0.784442E 04	0.000000E 00	0.000000E 00

SEMI-MAJOR AXES OF INTERCEPTOR, TARGET 0.64733E 07 0.67732E 07

TARGET TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.00000E 00	0.67475E 07	0.59033E 06	-0.66837E 03	0.76395E 04

CONTROL

S	SIG	Z	D	D0	COA	D2	CH0	CH2
0.40048E-03	0.40906E-03	-167.6241	79.3602	42.3021	2527.7813	37.0580	-0.2410	-4.1797
0.37091E-03	0.36731E-03	-0.0625	24.9720	13.2356	2462.8784	11.7363	1.1175	-1.0166
0.37711E-03	0.37355E-03	0.0002	24.9531	13.1952	2505.2168	11.7578	1.1294	-1.0513

TARGET TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.13195E 02	0.67379E 07	0.69106E 06	-0.78242E 03	0.76287E 04

INTERC TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.00000E 00	0.64733E 07	0.00000E 00	0.00000E 00	0.78444E 04
0.13195E 01	0.64732E 07	0.10356E 05	-0.84921E 01	0.78529E 04
0.26390E 01	0.64732E 07	0.20724E 05	-0.16975E 02	0.78615E 04
0.39585E 01	0.64732E 07	0.31103E 05	-0.25448E 02	0.78701E 04
0.52780E 01	0.64732E 07	0.41494E 05	-0.33912E 02	0.78786E 04
0.65976E 01	0.64731E 07	0.51895E 05	-0.42367E 02	0.78872E 04
0.79171E 01	0.64730E 07	0.62308E 05	-0.50812E 02	0.78957E 04
0.92366E 01	0.64730E 07	0.72733E 05	-0.59247E 02	0.79043E 04
0.10556E 02	0.64729E 07	0.83168E 05	-0.67673E 02	0.79129E 04
0.11875E 02	0.64728E 07	0.93615E 05	-0.76089E 02	0.79214E 04
0.13195E 02	0.64727E 07	0.10407E 06	-0.84494E 02	0.79300E 04

END OF FIRST BURN

TARGET TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.25301E 04	-0.66518E 07	0.12769E 07	-0.14457E 04	-0.75312E 04

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SECOND BURN

INTERC TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.25184E 04	-0.66326E 07	0.13590E 07	-0.15172E 04	-0.74292E 04
0.25195E 04	-0.66344E 07	0.13503E 07	-0.15069E 04	-0.74302E 04
0.25207E 04	-0.66361E 07	0.13415E 07	-0.14966E 04	-0.74312E 04
0.25219E 04	-0.66379E 07	0.13328E 07	-0.14863E 04	-0.74321E 04
0.25231E 04	-0.66396E 07	0.13241E 07	-0.14760E 04	-0.74331E 04
0.25242E 04	-0.66414E 07	0.13153E 07	-0.14656E 04	-0.74340E 04
0.25254E 04	-0.66431E 07	0.13066E 07	-0.14553E 04	-0.74349E 04
0.25266E 04	-0.66448E 07	0.12978E 07	-0.14450E 04	-0.74358E 04
0.25278E 04	-0.66465E 07	0.12891E 07	-0.14346E 04	-0.74367E 04
0.25289E 04	-0.66482E 07	0.12804E 07	-0.14243E 04	-0.74376E 04
0.25301E 04	-0.66498E 07	0.12716E 07	-0.14139E 04	-0.74385E 04

RENDEZVOUS

FIRST, SECOND BURN, MISSION	0.131952E 02	0.117578E 02	0.253016E 04
RELATIVE POSITION VECTOR	0.195103E 04	-0.526650E 04	
RELATIVE VELOCITY COMPONENTS	0.317404E 02	0.927441E 02	

CIRCULAR CASE, UPDATING AFTER 10 SECONDS

INPUT

NU	TAU	E	DT	0.398335E 15	0.583326E 03	0.418740E 04	0.100000E 02
RS	VS	GAS	DEL	0.677330E 07	0.766874E 04	0.000000E 00	0.872665E-01
R0	V0	GA0	PHI	0.647330E 07	0.784442E 04	0.000000E 00	0.000000E 00

SEMI-MAJOR AXES OF INTERCEPTOR, TARGET 0.64733E 07 0.67732E 07

TARGET TIME.	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.00000E 00	0.67475E 07	0.59033E 06	-0.66867E 03	0.76595E 04

CONTROL

S	SIG	Z	D	D0	LOA	D2	CH0	CH2
0.40048E-03	0.40906E-03	-167.6201	79.3602	42.3021	2527.7813	37.0580	-0.2410	-0.[illegible]797
0.37691E-03	0.36731E-03	-0.0625	24.9720	13.2356	2462.8784	11.7363	1.1175	-1.8166
0.37711E-03	0.37355E-03	0.0000	24.9531	13.1952	2505.2168	11.7578	1.1294	-1.8513

TARGET TIME.	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
-0.10000E 02	0.67404E 07	0.66668E 06	-0.75482E 03	0.76315E 04

INTERC TIME.	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.00000E 00	0.64733E 07	0.00000E 00	0.00000E 00	0.78444E 04
0.10000E 01	0.64732E 07	0.78476E 04	-0.64366E 01	0.78509E 04
0.20000E 01	0.64732E 07	0.15701E 05	-0.12867E 02	0.78574E 04
0.30000E 01	0.64732E 07	0.23562E 05	-0.19293E 02	0.78638E 04
0.39999E 01	0.64732E 07	0.31429E 05	-0.25714E 02	0.78703E 04
0.49999E 01	0.64732E 07	0.39303E 05	-0.32129E 02	0.78768E 04
0.59999E 01	0.64731E 07	0.47183E 05	-0.38539E 02	0.78833E 04
0.69999E 01	0.64731E 07	0.55069E 05	-0.44943E 02	0.78898E 04
0.79999E 01	0.64730E 07	0.62962E 05	-0.51342E 02	0.78963E 04
0.89999E 01	0.64730E 07	0.70862E 05	-0.57735E 02	0.79028E 04
0.99999E 01	0.64729E 07	0.78768E 05	-0.64123E 02	0.79093E 04

CONTROL

S	SIG	Z	D	D0	LOA	D2	CH0	CH2
0.37711E-03	0.37395E-03	-36.5366	25.4900	10.6402	2468.1411	14.8498	-0.[illegible]056	-0.[illegible]
0.37979E-03	0.37314E-03	0.0000	15.2585	5.2941	2512.1665	11.9644	1.1583	-1.8409

TARGET TIME.	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.13294E 02	0.67378E 07	0.69182E 06	-0.78328E 03	0.76286E 04

INTERC TIME.	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.99999E 01	0.64729E 07	0.78768E 05	-0.64123E 02	0.79093E 04
0.10329E 02	0.64729E 07	0.81374E 05	-0.66289E 02	0.79114E 04
0.10658E 02	0.64729E 07	0.83984E 05	-0.68455E 02	0.79136E 04
0.10988E 02	0.64728E 07	0.86587E 05	-0.70620E 02	0.79157E 04
0.11317E 02	0.64728E 07	0.89195E 05	-0.72784E 02	0.79179E 04
0.11647E 02	0.64728E 07	0.91804E 05	-0.74948E 02	0.79201E 04
0.11976E 02	0.64728E 07	0.94413E 05	-0.77111E 02	0.79222E 04
0.12305E 02	0.64727E 07	0.97023E 05	-0.79274E 02	0.79244E 04
0.12635E 02	0.64727E 07	0.99633E 05	-0.81436E 02	0.79266E 04
0.12964E 02	0.64727E 07	0.10224E 06	-0.83597E 02	0.79287E 04
0.13294E 02	0.64726E 07	0.10485E 06	-0.85758E 02	0.79309E 04

END OF FIRST BURN

TARGET TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.25374E 04	-0.66621E 07	0.12222E 07	-0.13838E 04	-0.75428E 04

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SECOND BURN

INTERP. TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.25254E 04	-0.66451E 07	0.13122E 07	-0.14631E 04	-0.74375E 04
0.25266E 04	-0.66468E 07	0.13033E 07	-0.14526E 04	-0.74384E 04
0.25278E 04	-0.66486E 07	0.12944E 07	-0.14421E 04	-0.74393E 04
0.25290E 04	-0.66503E 07	0.12855E 07	-0.14316E 04	-0.74402E 04
0.25302E 04	-0.66520E 07	0.12766E 07	-0.14211E 04	-0.74411E 04
0.25314E 04	-0.66537E 07	0.12677E 07	-0.14106E 04	-0.74420E 04
0.25326E 04	-0.66554E 07	0.12588E 07	-0.14001E 04	-0.74428E 04
0.25338E 04	-0.66571E 07	0.12499E 07	-0.13896E 04	-0.74437E 04
0.25350E 04	-0.66587E 07	0.12410E 07	-0.13791E 04	-0.74445E 04
0.25362E 04	-0.66604E 07	0.12321E 07	-0.13686E 04	-0.74453E 04
0.25374E 04	-0.66620E 07	0.12232E 07	-0.13584E 04	-0.74461E 04

RENDEZVOUS

FIRST, SECOND BURN, MISSION	0.132940E 02	0.119643E 02	0.253741E 04
RELATIVE POSITION VECTOR	0.580604E 02	0.959250E 03	
RELATIVE VELOCITY COMPONENTS	0.25[illegible]16E 02	0.956972E 02	

ELLIPTIC CASE, UPDATING EVERY 10 SECONDS

INPUT

MU	TAU	E	DT	0.398335E 15	0.583326E 03	0.418740E 04	0.100000E 02
R3	V3	GA3	DEL	0.680000E 07	0.765000E 04	-0.999994E-02	0.150000E 00
R0	V0	GA0	PHI	0.640000E 07	0.780000E 04	0.999999E-02	0.000000E 00

SEMI-MAJOR AXES OF INTERCEPTOR, TARGET 0.62592E 07 0.67934E 07

TARGET TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.00000E 00	0.67236E 07	0.10161E 07	-0.12187E 04	0.75522E 04

CONTROL

S	SIG	Z	D	D0	COA	D2	CH0	CH2
0.39380E-03	0.41027E-03	-254.6309	145.8677	90.5155	2368.9336	55.3522	2.8885	-0.0603
0.39522E-03	0.39254E-03	-0.0244	50.1244	34.9603	2599.0747	15.1641	2.2648	-1.4558
0.40289E-03	0.40033E-03	0.1196	50.1757	35.0400	2651.9336	15.1356	2.2687	-1.5063
0.39532E-03	0.39265E-03	-0.0415	50.1275	34.9642	2599.7549	15.1633	2.2650	-1.4562
0.39545E-03	0.39280E-03	-0.0010	50.1203	34.9592	2600.8369	15.1611	2.2648	-1.4579
0.39546E-03	0.39278E-03	-0.0005	50.1204	34.9590	2600.7295	15.1614	2.2648	-1.4578

TARGET TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.10000E 02	0.67110E 07	0.10916E 07	-0.13038E 04	0.75389E 04

INTERC TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.00000E 00	0.64000E 07	0.00000E 00	0.77998E 02	0.77996E 04
0.10000E 01	0.64000E 07	0.78023E 04	0.63678E 02	0.78051E 04
0.20000E 01	0.64001E 07	0.15610E 05	0.49350E 02	0.78106E 04
0.30000E 01	0.64001E 07	0.23423E 05	0.35014E 02	0.78161E 04
0.39999E 01	0.64001E 07	0.31242E 05	0.20671E 02	0.78216E 04
0.49999E 01	0.64002E 07	0.39066E 05	0.63203E 01	0.78271E 04
0.59999E 01	0.64001E 07	0.46896E 05	-0.80385E 01	0.78326E 04
0.69999E 01	0.64001E 07	0.54732E 05	-0.22405E 02	0.78381E 04
0.79999E 01	0.64001E 07	0.62573E 05	-0.36779E 02	0.78436E 04
0.89999E 01	0.64001E 07	0.70419E 05	-0.51162E 02	0.78491E 04
0.99999E 01	0.64000E 07	0.78271E 05	-0.65552E 02	0.78546E 04

CONTROL

S	SIG	Z	D	D0	COA	D2	CH0	CH2
0.39546E-03	0.39278E-03	-33.8467	49.7338	31.6734	2567.2373	18.0604	2.5155	-0.8844
0.39883E-03	0.39362E-03	-0.0015	40.7866	25.3889	2615.7324	15.3977	2.2748	-1.4419
0.39866E-03	0.39345E-03	0.0005	40.7864	25.3868	2614.5205	15.3996	2.2747	-1.4408

TARGET TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.20000E 02	0.66975E 07	0.11669E 07	-0.13888E 04	0.75246E 04

INTERC TIME,	CARTESIAN COORDINATES	AND	VELOCITY COMPONENTS	
0.99999E 01	0.64000E 07	0.78271E 05	-0.65552E 02	0.78546E 04
0.10999E 02	0.63999E 07	0.86128E 05	-0.80005E 02	0.78601E 04
0.11999E 02	0.63998E 07	0.93991E 05	-0.94467E 02	0.78655E 04
0.12999E 02	0.63997E 07	0.10185E 06	-0.10893E 03	0.78709E 04
0.13999E 02	0.63996E 07	0.10973E 06	-0.12341E 03	0.78764E 04
0.14999E 02	0.63995E 07	0.11761E 06	-0.13790E 03	0.78818E 04
0.15999E 02	0.63993E 07	0.12549E 06	-0.15239E 03	0.78873E 04
0.16999E 02	0.63992E 07	0.13338E 06	-0.16689E 03	0.78927E 04
0.17999E 02	0.63990E 07	0.14128E 06	-0.18140E 03	0.78981E 04
0.18999E 02	0.63988E 07	0.14918E 06	-0.19592E 03	0.79035E 04
0.19999E 02	0.63986E 07	0.15708E 06	-0.21045E 03	0.79090E 04

CONTROL

S	SIG	Z	D	D0	COA	D2	CH0	CH2
0.39866E-03	0.39345E-03	-35.8730	41.5457	22.9310	2577.8579	18.6146	2.6451	-0.5430
0.40177E-03	0.39384E-03	-0.2093	31.3362	15.6655	2626.6616	15.6407	2.2976	-1.4248
0.40201E-03	0.39590E-03	-24.4316	37.3389	20.3197	2610.1743	17.0192	2.5654	-1.4105
0.40185E-03	0.39392E-03	-0.0215	31.3088	15.6685	2627.1577	15.6404	2.2979	-1.4250
0.40191E-03	0.39398E-03	-0.0024	31.3056	15.6662	2627.2679	15.6394	2.2977	-1.4257

TARGET TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.30000E 02	0.66852E 07	0.12421E 07	-0.14736E 04	0.75093E 04

INTERC TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.19999E 02	0.63986E 07	0.15708E 06	-0.21045E 03	0.79090E 04
0.20999E 02	0.63984E 07	0.16500E 06	-0.22511E 03	0.79143E 04
0.21999E 02	0.63981E 07	0.17291E 06	-0.23979E 03	0.79196E 04
0.22999E 02	0.63979E 07	0.18084E 06	-0.25447E 03	0.79249E 04
0.23999E 02	0.63976E 07	0.18876E 06	-0.26916E 03	0.79302E 04
0.24999E 02	0.63974E 07	0.19670E 06	-0.28386E 03	0.79355E 04
0.25999E 02	0.63971E 07	0.20463E 06	-0.29856E 03	0.79408E 04
0.26999E 02	0.63967E 07	0.21256E 06	-0.31328E 03	0.79461E 04
0.27999E 02	0.63964E 07	0.22053E 06	-0.32801E 03	0.79514E 04
0.28999E 02	0.63961E 07	0.22848E 06	-0.34274E 03	0.79567E 04
0.29999E 02	0.63957E 07	0.23644E 06	-0.35748E 03	0.79620E 04

CONTROL

S	SIG	/	D	D0	COA	U2	CH0	CH2
0.40191E-03	0.39398E-03	-37.5737	33.9510	14.8405	2587.3291	19.1105	2.9042	-0.5075
0.40445E-03	0.39364E-03	-0.0010	21.7150	5.8279	2634.8843	15.8871	2.3739	-1.4085
0.40350E-03	0.39270E-03	0.0010	21.7160	5.8167	2628.4478	15.8994	2.3707	-1.4026
0.40348E-03	0.39268E-03	0.0000	21.7160	5.8164	2628.3174	15.8997	2.3707	-1.4025

TARGET TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.35816E 02	0.66745E 07	0.12857E 07	-0.15229E 04	0.75000E 04

INTERC TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.29999E 02	0.63957E 07	0.23644E 06	-0.35748E 03	0.79620E 04
0.30581E 02	0.63955E 07	0.24107E 06	-0.36629E 03	0.79649E 04
0.31163E 02	0.63953E 07	0.24570E 06	-0.37511E 03	0.79677E 04
0.31744E 02	0.63951E 07	0.25034E 06	-0.38393E 03	0.79706E 04
0.32326E 02	0.63949E 07	0.25498E 06	-0.39275E 03	0.79734E 04
0.32908E 02	0.63946E 07	0.25961E 06	-0.40157E 03	0.79763E 04
0.33489E 02	0.63944E 07	0.26425E 06	-0.41040E 03	0.79791E 04
0.34071E 02	0.63941E 07	0.26890E 06	-0.41923E 03	0.79820E 04
0.34652E 02	0.63939E 07	0.27354E 06	-0.42806E 03	0.79848E 04
0.35234E 02	0.63936E 07	0.27818E 06	-0.43690E 03	0.79877E 04
0.35816E 02	0.63934E 07	0.28283E 06	-0.44574E 03	0.79905E 04

END OF FIRST BURN

TARGET TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.26800E 04	-0.67640E 07	-0.46714E 06	0.45317E 03	-0.76590E 04

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SECOND BURN

INTERC TIME,	CARTESIAN COORDINATES		AND VELOCITY COMPONENTS	
0.26641E 04	-0.67721E 07	-0.33832E 06	0.28609E 03	-0.75480E 04
0.26657E 04	-0.67716E 07	-0.35032E 06	0.29961E 03	-0.75459E 04
0.26673E 04	-0.67711E 07	-0.36232E 06	0.31313E 03	-0.75438E 04
0.26689E 04	-0.67706E 07	-0.37431E 06	0.32665E 03	-0.75417E 04
0.26704E 04	-0.67701E 07	-0.38630E 06	0.34016E 03	-0.75395E 04
0.26720E 04	-0.67696E 07	-0.39829E 06	0.35367E 03	-0.75373E 04
0.26736E 04	-0.67690E 07	-0.41027E 06	0.36719E 03	-0.75351E 04
0.26752E 04	-0.67684E 07	-0.42225E 06	0.38070E 03	-0.75329E 04
0.26768E 04	-0.67678E 07	-0.43422E 06	0.39420E 03	-0.75306E 04
0.26784E 04	-0.67671E 07	-0.44620E 06	0.40771E 03	-0.75283E 04
0.26800E 04	-0.67665E 07	-0.45816E 06	0.42121E 03	-0.75260E 04

RENDEZVOUS

FIRST, SECOND BURN, MISSION	0.358162E 02	0.158996E 02	0.268002E 04
RELATIVE POSITION VECTOR	-0.250600E 04	0.897331E 04	
RELATIVE VELOCITY COMPONENTS	-0.319635E 02	0.132915E 03	

Section 8
CONCLUSIONS

Planning an optimal rendezvous and steering the intercepting vehicle is a complex problem of calculus of variations. Even a modern computer takes too long for solving such problems on a real time basis.

For this purpose simplifying assumptions have to be introduced, and a trade-off between simplicity of the guidance equations and accuracy of the results has to be made. The simplest way, the impulse approximation, turns out to be insufficient in accuracy for realistic cases.

The present approach is very successful in coming up with rather simple guidance equations due to the use of Levi-Civita's variables. The accuracy is such that a good closed-loop terminal guidance scheme could take over after the coast phase.

However, when larger and faster on-board computers are available, it might be worthwhile to seek more sophisticated guidance schemes which allow a more precise and a more economical steering of the interceptor.

Section 9
REFERENCES

1. Pontryagin, L. S., et al., "The Mathematical Theory of Optimal Processes," Wiley, New York, 1962.

2. Trautwein, W. and J. Waldvogel, "A Simple Adaptive Guidance Scheme for Minimum Fuel Two-Finite-Burns Rendezvous," AIAA Guidance, Control, and Flight Dynamics Conference, Pasadena, 1968, Paper No. 68-858.

3. Trautwein, W. and J. Waldvogel, "Optimum Rendezvous Guidance Study," Interim Report, Contract NAS8-21146, LMSC-HREC A791387, Lockheed Missiles & Space Company, Huntsville, Ala., June 1968.

4. Kliger, I. E., "Rendezvous Guidance Study," Final Report, Contract NAS8-18036, LMSC/HREC A783890, Lockheed Missiles & Space Company, Huntsville, Ala., January 1967.

5. Trautwein, W., "Extension of the Dual-Phase Plane Terminal Guidance Method to Long Burn Times," Technical Note 54/30-13, LMSC/HREC A791235, February 1968.

6. Burdet, C. A., "Intermediaere Bahn im Gravitationsfeld der Erde," Inst. for Operations Research, Swiss Federal Institute of Technology, September 1968 (in publication).

7. Levi-Civita, T., "Sur la Résolution Qualitative du Problème Restreint des Trois Corps," Opere Mathematiche, Vol. 2, Bologna 1956.

8. Kustaanheimo, P. and E. Stiefel, "Perturbation Theory of Kepler Motion Based on Spinor Regularization," Journal für die reine und angewandte Mathematik, 218 (1965), 204-219.

9. Stiefel, E., et al., "Methods of Regularization for Computing Orbits in Celestial Mechanics," NASA CR-769, June 1967.

APPENDIX

COMPUTER PROGRAMS

APPENDIX
COMPUTER PROGRAMS

Prior to the program listings there follows a table giving the meaning of the most important program variables or the corresponding name in this report. The input the simulation program requires can be seen in the flow chart of Section 7, and the output is self-explanatory.

Main Program

DT	updating time interval
NSTP	number of Runge-Kutta steps in one updating time interval
SCH	integration step
TINT	current updating time interval
UPD =	TRUE during updating period
X0, Y0	initial guesses for s, σ
XT, YT	Cartesian coordinates of the target
UT, VT	velocity components of the target
X(1)	time
X(2), X(3)	Cartesian coordinates of the interceptor
X(4), X(5)	velocity components of the interceptor

A0	a_0	D0	D_0	R0	r_0
A2	a_2	D2	D_2	R2	r_2
A10	α_{10}	DA1	$\Delta\alpha_1$	R3	r_3
A20	α_{20}	DA2	$\Delta\alpha_2$	TAR	t_{tar}
A13	α_{13}	DEL	δ	TAU	τ
A23	α_{23}	DG1	$\Delta\gamma_1$	U12	u_{12}
B10	β_{10}	DG2	$\Delta\gamma_2$	U22	u_{22}
B20	β_{20}	E	e	V0	v_0
B13	β_{13}	GA0	γ_0	V3	v_3

B23	β_{23}	GA3	γ_3	W12	w_{12}
CH0	χ_0	MU	μ	W22	w_{22}
CH2	χ_2	OM0	ω_0	Z	Z
COA	t_{coa}	OM2	ω_2		
D	D	PHI	φ		

Subprogram CONTRL

H	initial step size
NI	counts iteration cycles
TOL	tolerance for stopping the iteration
YOLD	old approximation for σ in secant method

E1	ϵ_1	X0	s_0	Y0	$\sigma_0{}^*$
E2	ϵ_2	X1	s_1	Y1	$\sigma_1{}^*$
		X2	s_2	Y2	$\sigma_2{}^*$

Subprogram KEPLER

A	a	DT	Δt	U2	u_2
A1	α_1	OM	ω	V	$\dot{x}_2$
A2	α_2	R	r	W1	w_1
B1	β_1	SIG	σ	W2	w_2
B2	β_2	U	$\dot{x}_1$	X	x_1
		U1	u_1	Y	x_2

```
C     MAIN PROGRAM FOR RENDEZVOUS SIMULATION
      REAL MU
      LOGICAL UPD
      COMMON/PARAM / MU,TAU,E,COA,CH0,CH2
      COMMON/OTHERS/ V0,V3,GA0,GA3,DEL,PHI
      COMMON/TRANSF/ X0,Y0,    A0,A2,OM0,OM2,R0,R2,R3,D,D0,D2,
     1              DG1,DG2,DA1,DA2,DES,DET,TAR,Z,SS0,SS2,
     2              A10,A20,B10,B20,A13,A23,B13,B23,U12,U22,W12,W22
      COMMON/RKG   / X(5),DX(5)
      READ (5,1) MU,TAU,E,DT,R3,V3,GA3,DEL,R0,V0,GA0,PHI
    1 FORMAT(4E16.6)
      WRITE(6,7) MU,TAU,E,DT,R3,V3,GA3,DEL,R0,V0,GA0,PHI
    7 FORMAT( 6H1INPUT/,
     1        20H MU   TAU   E    DT   , 4E13.7/,
     2        20H R3   V3   GA3  DEL   , 4E13.7/,
     3        20H R0   V0   GA0  PHI   , 4E13.7/)
C
      NSTP = 10
      TT   = 0.
      X(1) = 0.
      X(2) = R0 * COS(PHI)
      X(3) = R0 * SIN(PHI)
      X(4) = V0 * SIN(GA0-PHI)
      X(5) = V0 * COS(GA0-PHI)
      A0   = 1./(2./R0 - V0*V0/MU)
      X0 = 3.1415926536 * SQRT(A0/MU)
      A2   = 1./(2./R3 - V3*V3/MU)
      Y0 = 3.1415926536 * SQRT(A2/MU)
      WRITE(6, 8) A0,A2
    8 FORMAT(///39H SEMI-MAJOR AXES OF  INTERCEPTOR, TARGET, 2E13.7)
      TINT = DT
      UPD = .TRUE.
      XT = R3 * COS(DEL)
      YT = R3 * SIN(DEL)
      UT = V3 * SIN(GA3-DEL)
      VT = V3 * COS(GA3-DEL)
      WRITE(6, 3) TT,XT,YT,UT,VT
    3 FORMAT(//69H TARGET TIME,        CARTESIAN COORDINATES   AND    VELO
     1CITY COMPONENTS/,
     2       F13.7, 2(4X,2F13.7))
   60 CALL CONTRL
      IF (TINT .LE. D0) GO TO 61
      TINT = D0
      UPD =.FALSE.
   61 CALL KEPLER(TINT,XT,YT,UT,VT)
      TT = TT + TINT
      WRITE(6, 3) TT,XT,YT,UT,VT
      WRITE(6,10)
   10 FORMAT(//69H INTERC TIME,        CARTESIAN COORDINATES   AND    VELO
     1CITY COMPONENTS/)
      CALL INTGRT(0.0)
      WRITE(6,2) X(1),X(2),X(3),X(4),X(5)
    2 FORMAT(E13.7, 2(4X,2E13.7))
      SCH = TINT / FLOAT(NSTP)
      DO 62 N=1,NSTP
      CALL INTGRT(SCH)
   62 WRITE(6,2) X(1),X(2),X(3),X(4),X(5)
      R0 = SQRT(X(2)**2 + X(3)**2)
      PHI= ATAN2(X(3),X(2))
      V0 = SQRT(X(4)**2 + X(5)**2)
```

```
      GA0 = PHI + 1.5707963268 - ATAN2(X(5),X(4))
      R3 = SQRT(XT*XT + YT*YT)
      DEL = ATAN2(YT,XT)
      V3 = SQRT(UT*UT + VT*VT)
      GA3= DEL + 1.5707963268 - ATAN2(VT,UT)
      IF (UPD) GO TO 60
      WRITE(6,11)
   11 FORMAT(  /18H END OF FIRST BURN)
      CALL KEPLER(COA+D2,XT,YT,UT,VT)
      TT = TT + COA + D2
      WRITE(6, 3) TT,XT,YT,UT,VT
      WRITE(6,12)
   12 FORMAT(  /11H RENDEZVOUS)
      CALL KEPLER(COA,X(2),X(3),X(4),X(5))
      D0 = X(1)
      X(1) = X(1) + COA
      CH0 = CH2
      WRITE(6,13)
   13 FORMAT(///12H SECOND BURN)
      WRITE(6,10)
      CALL INTGRT(0.0)
      WRITE(6,2) X(1),X(2),X(3),X(4),X(5)
      SCH = D2/FLOAT(NSTP)
      DO 63 N =1,NSTP
      CALL INTGRT(SCH)
   63 WRITE(6,2) X(1),X(2),X(3),X(4),X(5)
      WRITE(6,12)
      TE2 = X(2)-XT
      TE3 = X(3)-YT
      TE4 = X(4)-UT
      TE5 = X(5)-VT
      WRITE(6,14) D0,D2,X(1),TE2,TE3,TE4,TE5
   14 FORMAT(////29H FIRST, SECOND BURN, MISSION , 3E13.7/,
     1           29H RELATIVE POSITION VECTOR     , 2E13.7/,
     2           29H RELATIVE VELOCITY COMPONENTS, 2E13.7)
      STOP
      END
```

```
      SUBROUTINE CONTRL
      REAL MU
      LOGICAL FLAG
      COMMON/PARAM / MU,TAU,E,COA,CHC,CH2
      COMMON/OTHERS/ V0,V3,GA0,GA3,DEL,PHI
      COMMON/TRANSF/ X0,Y0,    A0,A2,OM0,OM2,R0,R2,R3,D,D0,D2,
     1               DG1,DG2,DA1,DA2,DES,DET,IAR,Z,S50,S52,
     2               A10,A20,B10,B20,A13,A23,B13,B23,U12,U22,W12,W22
      A0   = 1./(2./R0 - V0*V0/MU)
      OM0  = SQRT(.25*MU/A0)
      A2   = 1./(2./R3 - V3*V3/MU)
      OM2  = SQRT(.25*MU/A2)
      DE2  = .5 * PHI
      TEMP= SQRT(R0)
      A10  = TEMP * COS(DE2)
      A20  = TEMP * SIN(DE2)
      DE2  = GA0 - DE2
      TEMP=.5*TEMP*V0/OM0
      B10  = TEMP * SIN(DE2)
      B20  = TEMP * COS(DE2)
      DE2  = DEL/2.
      TEMP= SQRT(R3)
      A13  = TEMP * COS(DE2)
      A23  = TEMP * SIN(DE2)
      DE2  = GA3 - DE2
      TEMP= .5*TEMP*V3/OM2
      B13  = TEMP * SIN(DE2)
      B23  = TEMP * COS(DE2)

      H    = .03* X0
      E1   = .0012
      E2   = .150
      TOL= 1.00E-6
      WRITE(6, 4)
      FORMAT(////8H CONTROL/, 6X,1HS,11X3HSIG,10X,1HZ,9X,1HD,8X,2HD0,
     1       8X,3HCOA,7X,2HD2,7X,3HCH0,5X,3HCH2 )
      NI=0
      STEP= H
      FLAG= .FALSE.
      Y    = Y0
      CALL FCT(X0,Y)
      TC1  = R0*DG2 - B10*DET
      TC2  = R0*DG1 + B20*DET
      CH0  = ATAN2(TC1,TC2)
      TC1  = R2*DA2 - W12*DES
      TC2  = R2*DA1 + W22*DES
      CH2  = ATAN2(TC1,TC2) + ATAN2(U22,U12)
      WRITE(6, 6) X0,Y,Z,D,D0,COA,D2,CH0,CH2
      FORMAT(2F13.5, 5F10.4, 2F8.4)
      NI=NI+1
      IF(NI .LE.  7) GO TO 43
      WRITE(6,44)
      FORMAT(18H 7 ITERATION STEPS//)
      GO TO 50
      ZC   =    E1    * ABS(Z)
      IF ( ZC    .LT.    TOL   ) GO TO 50
      IF (FLAG) STEP = E*Z
      DO 71  I=1,7
      ZOLD= Z
```

```
      YOLD= Y
      Y    = YOLD + STEP
      CALL FCT(X0,Y)
      IF (FLAG) GO TO 83
      F    = H / (ZOLD-Z)
      FLAG= .TRUE.
   83 IF (Z .EQ. ZOLD) GO TO 80
      IF (ABS(Z) .GT. ZC) GO TO 72
   80 Y0   = Y
      C0   = D
      GO TO 73
   72 STEP= -Z / (ZOLD-Z)*(YOLD-Y)
   71 CONTINUE
   73 X1   = X0 + H
      Y    = Y0 + H
      CALL FCT(X1,Y)
      STEP= F * Z
      DO 74 I=1,7
      ZOLD= Z
      YOLD= Y
      Y    = YOLD + STEP
      CALL FCT(X1,Y)
      IF (Z .EQ. ZOLD) GO TO 81
      IF (ABS(Z) .GT. ZC) GO TO 75
   81 Y1   = Y
      C1   = D
      GO TO 76
   75 STEP= -Z / (ZOLD-Z)*(YOLD-Y)
   74 CONTINUE
   76 X2   = X0 - H
      Y    = 2.*Y0 - Y1
      CALL FCT(X2,Y)
      STEP= F * Z
      DO 77 I=1,7
      ZOLD= Z
      YOLD= Y
      Y    = YOLD + STEP
      CALL FCT(X2,Y)
      IF (Z .EQ. ZOLD) GO TO 82
      IF (ABS(Z) .GT. ZC) GO TO 78
   82 Y2   = Y
      C2   = D
      GO TO 79
   78 STEP= -Z / (ZOLD-Z)*(YOLD-Y)
   77 CONTINUE
   79 STP  = -.5*(C1-C2) / (C1-2.*C0+C2)
      X0   = X0 + STP*H
      Y0   = Y0 + .5*STP*(Y1-Y2+STP*(Y1-2.*Y0+Y2))
      H    =  E2 *STP*H
      GO TO 70
   50 Y0   = Y
      RETURN
      END
```

```
      SUBROUTINE  FCT(S,SIG)
      REAL MU,LAM
      COMMON/PARAM / MU,TAU,E,COA,CHO,CH2
      COMMON/TRANSF/ XO,YO,   AO,A2,OMO,OM2,RO,R2,R3,D,DO,D2,
     1               DG1,DG2,DA1,DA2,DES,DET,TAR,Z,SSO,SS2,
     2               A10,A20,B10,B20,A13,A23,B13,B23,U12,U22,W12,W22
      CO   = COS(OMO*S)
      SO   = SIN(OMO*S)
      C2   = COS(OM2*SIG)
      S2   = SIN(OM2*SIG)
      U12  = C2*A13 + S2*B13
      U22  = C2*A23 + S2*B23
      W12  =-S2*A13 + C2*B13
      W22  =-S2*A23 + C2*B23
      A12  = CO*U12 - SO*W12
      A22  = CO*U22 - SO*W22
      B12  = SO*U12 + CO*W12
      B22  = SO*U22 + CO*W22
      R2   = U12*U12 + U22*U22
      DA1  = A12 - A10
      DA2  = A22 - A20
      DG1  = DA1*CO + (B12-B10)*SO
      DG2  = DA2*CO + (B22-B20)*SO
      DET  = DG1*B20 - DG2*B10
      DES  = DA1*W22 - DA2*W12
      GO   = OMO**3 * SQRT(RO*(DG1*DG1+DG2*DG2) + (1.+AO/RO)*DET*DET)
      G2   = OM2**3 * SQRT(R2*(DA1*DA1+DA2*DA2) + (1.+A2/R2)*DES*DES)
      LAM  = .25*MU*E*SO
      TEMP= LAM + GO/2.
      DO   = TAU*GO / TEMP
      D    = TAU * (LAM/(LAM+.5*G2)) * ((GO+G2)/TEMP)
      D2   = D - DO
      B11  = B12 + DA1*CO/SO
      B21  = B22 + DA2*CO/SO
      A1   = (RO+B11*B11+B21*B21)/2.
      OM1  = SQRT(.25*MU/A1)
      SSO  = DO/RO
      SS2  = (OM2*D2)/(OMO*R2)
      TEMP= 2.*OM1*(S-SS2)
      TEM1= 2.*OM1*SSO
      COA  = OMO/OM1 * (A1*(S-SSO-SS2)
     1      + ((A10*B11+A20*B21)*(COS(TEM1)-COS(TEMP))
     2      + (RO-A1) * (SIN(TEMP)-SIN(TEM1))) / (2.*OM1))
      TAR  = A2*SIG + ((A13*B13+A23*B23)*S2 + (R3-A2)*C2) * S2/OM2
      Z    = COA + D - TAR
      RETURN
      END
```

```
      SUBROUTINE INTGRT (H)
CRKG  THIS SUBROUTINE INTEGRATES FROM POINTS Y(T) TO THE POINT Y(T+H).
C     THE DERIVATIVES ARE ALSO COMPUTED AT T+H.  THE ROUTINE REQUIRES
C     THAT N, Y, DY BE IN COMMON, AND THAT Y AND DY BE DIMENSIONED
C     BY N.  TO INITIALIZE CALL WITH H=0.
C     ENTERING THIS ROUTINE WITH H=0 CAUSES THE CONSTANTS TO BE SET,
C     THE J VECTOR SET TO ZERO AND THE DERIVATIVES RECOMPUTED VIA THE
C     SUBROUTINE CALLED DEQ
C                             COMMON REGION
      COMMON/RKG    / X(5),DX(5)
      DIMENSION A(4), B(4), C(4), Q(33)
      IF ( H .EQ. 0.0) GO TO 4
    1 DO 3 I= 1,4
      DO 2 J=1,5
      RRR=A(I)*DX(J)-B(I)*Q(J)
      Q(J)=Q(J)+3.0*RRR-C(I)*DX(J)
    2 X(J)=X(J)+H*RRR
    3 CALL DEQ
      RETURN
    4 D= .707106781E+00
      A(1)= 0.5
      A(2)= 1.0-D
      A(3)= 1.0+D
      A(4)= 1.0 / 6.0
      B(1)= 1.0
      B(2)= A(2)
      B(3)= A(3)
      B(4)= 1.0 / 3.0
      C(1)= A(1)
      C(2)= A(2)
      C(3)= A(3)
      C(4)= 0.5
      DO 5 J=1,32
    5 Q(J)= 0.0
      CALL DEQ
      DX(1)=1.0
      RETURN
      END
```

```
SUBROUTINE DEQ
REAL MU
COMMON/PARAM / MU,TAU,E,COA,CH0,CH2
COMMON/RKG    / X(5),DX(5)
R  = SQRT(X(2)**2 + X(3)**2)
R  = -MU/(R*R*R)
S  = TAU - X(1)
IF (X(1) .GT. 500.)   S = S-COA
S  = E / S
DX(2) = X(4)
DX(3) = X(5)
DX(4) = R*X(2) + S*COS(CH0)
DX(5) = R*X(3) + S*SIN(CH0)
RETURN
END
```

```
SUBROUTINE  KEPLER(DT,X,Y,U,V)
X,Y,U,V ARE THE CARTESIAN COORDINATES AND VELOCITY COMPONENTS OF
THE VEHICLE. THEY ARE INPUT AS WELL AS OUTPUT PARAMETERS. DT IS
THE TIME INCREMENT DURING THE KEPLERIAN MOTION. THE GRAVITATIONAL
PARAMETER MU OF THE CENTRAL BODY MUST BE IN COMMON.
REAL MU
COMMON/PARAM / MU,TAU,E,COA,CH0
R  = SQRT(X*X+Y*Y)
A  = 1. / (2./R - (U*U+V*V)/MU)
OM = SQRT(.25*MU/A)
A1 = SQRT(.5*(R+ABS(X)))
A2 = .5*Y/A1
IF (X .GT. .0) GO TO 11
B  = A1
A1 = A2
A2 = B
B1 = .5*(A1*U + A2*V)/OM
B2 = .5*(A1*V - A2*U)/OM
QC = R-A
QS =  A1*B1 + A2*B2
SIG= (DT - .5*QS/OM)/A
DO 12 I=1,7
S  = SIN(OM*SIG)
C  = COS(OM*SIG)
F  = A*SIG + S*(QC*C+QS*S)/OM - DT
FP = A + QC*(C*C-S*S) + 2.*QS*C*S
SG = F / FP
IF (ABS(SG) .LT. 1.5E-10) GO TO 13
SIG= SIG - SG
U1 = A1*C + B1*S
U2 = A2*C + B2*S
W1 = B1*C - A1*S
W2 = B2*C - A2*S
X  = U1*U1 - U2*U2
Y  = 2.*U1*U2
B  = OM / (.5*X+U2*U2)
U  = B*(U1*W1 - U2*W2)
V  = B*(U2*W1 + U1*W2)
RETURN
END
```